8th Grade

1. In triangle ABC, point O is the circumcenter. The line AO intersects BC at point T, and the perpendiculars drawn from T to AB and AC intersect the radii OB and OC at points E and F, respectively. Prove that $BE = CF$.

(*Oleksii Karlyuchenko*)

Solution. Let AD be the diameter of the circumcircle of triangle ABC (Fig. 1). Then $BD \perp AB$ and $TE \perp AB$, which implies $TE \parallel BD$. Triangle OBD is isosceles ($OB = OD$ as radii). If $\angle OBD = \angle ODB = \alpha$, then $\angle OET = \angle OTE = \alpha$, so $OE = OT$ and $BE = OB - OE = OD - OT = DT$. Similarly, $CF = DT$, hence $BE = CF$.

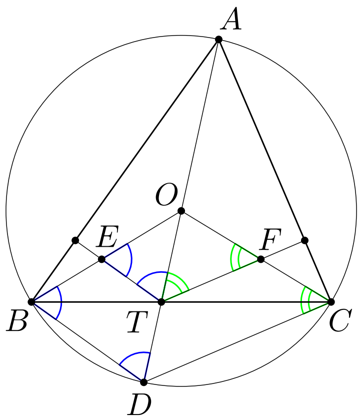


Fig. 1

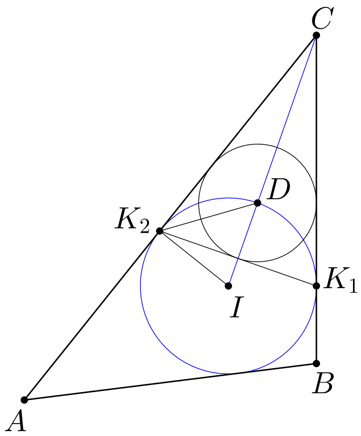


Fig. 2

2. Given a triangle ABC, with a marked point I as its incenter, and K_1 and K_2 being the points of tangency of the incircle with sides BC and AC, respectively. Using a compass and a ruler, construct the incenter of triangle CK_1K_2 with the minimal possible number of lines (a line is a straight line or a circle).

(*Hryhorii Filippovskyi*)

Solution. Clearly, that a single line is insufficient for the construction. Draw two lines: the incircle of triangle ABC (with center I and radius IK_1) and the segment CI. Let these intersect at point D (Fig. 2). We will show that D is the incenter of triangle CK_1K_2. Indeed, D lies on the angle bisector of $\angle C$, so it suffices to prove that K_2D is the angle bisector of $\angle CK_2K_1$. Let $\angle ACB = 2\alpha$. Since CI is the angle bisector, triangle CIK_2 is right-angled, and triangles K_2ID and CK_1K_2 are isosceles, step by step, we find

$$\angle CIK_2 = 90^\circ - \alpha,\ \angle IK_2D = 90^\circ - \tfrac{1}{2}\angle CIK_2 = 45^\circ + \tfrac{\alpha}{2},$$

$$\angle CK_2D = 90^\circ - \angle IK_2D = 45^\circ - \tfrac{\alpha}{2} = \tfrac{1}{2}\angle CK_2K_1,$$

which completing the proof.

3. Let ABC be a right triangle ($\angle C = 90°$), N be the midpoint of arc BAC of the circumcircle, and K the intersection point of CN with AB. On the extension of AK beyond K, let T be the point chosen such that $TK = KA$. Prove that the circle with center T and radius TK is tangent to BC.

(*Mykhailo Sydorenko*)

Solution.

Draw $TD \perp BC$ and $TE \perp AC$ (Fig. 3). It suffices to prove that $TD = TK$. Since KE is the median of the right triangle AET, drawn to the hypotenuse, we have $KE = TK = KA$, and since $TDCE$ is a rectangle, $TD = CE$. It remains to prove that $KE = CE$. Let $\angle BAC = 2\alpha$. Then $\angle BNC = 2\alpha$, implying $\angle BCN = \angle CBN = 90° - \alpha$, $\angle KCE = 90° - \angle BCN = \alpha$, $\angle KEA = 2\alpha$ and $\angle CKE = \angle KEA - \angle KCE = \alpha = \angle KCE$. Thus, triangle KEC is isosceles, completing the proof.

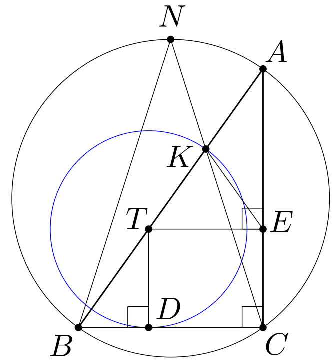


Fig. 3

4. Let ABC be an acute triangle, AD, BE, and CF its altitudes, and H the orthocenter. On the rays AD, BE, and CF, points A_1, B_1, and C_1 chosen such that $AA_1 = HD$, $BB_1 = HE$, and $CC_1 = HF$ respectively. Let A_2, B_2, and C_2 be the midpoints of A_1D, B_1E, and C_1F, respectively. Prove that the points H, A_2, B_2, and C_2 lie on the same circle.

(*Mykhailo Barkulov*)

Solution.

Let O be the circumcenter of triangle ABC. Extend the altitudes and mark points H_1, H_2, and H_3 such that $DH_1 = DH$, $EH_2 = EH$, and $FH_1 = FH$. (Fig. 4). Then A_2 is the midpoint of both segments A_1D and AH_1. The right triangles BDH and BDH_1 are congruent by two legs. Hence $\angle BH_1A = \angle BHD = 90° - \angle HBC = \angle BCA$, so point H_1 lies on the circumcircle of triangle ABC. Therefore, the perpendicular bisector of AH_1 passes through O, implying $\angle HA_2O = 90°$. Similarly, $\angle HB_2O = 90°$ and $\angle HC_2O = 90°$. Thus, the points H, A_2, B_2, C_2, and O lie on a circle with diameter HO.

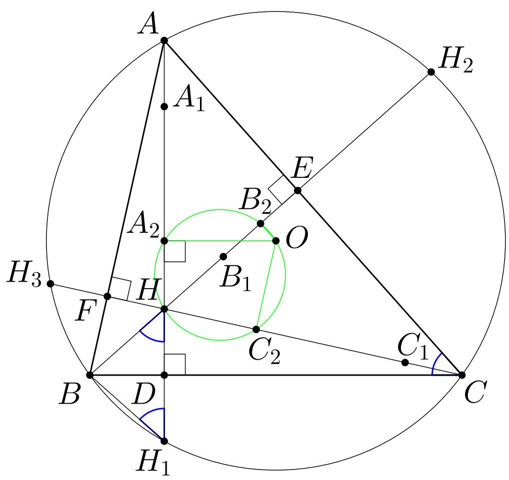
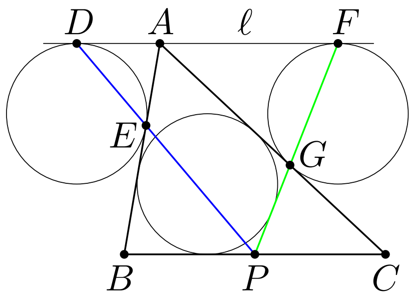


Fig. 4

5. Through vertex A of triangle ABC, a line $\ell \parallel BC$ is drawn. Two circles, each congruent to the incircle of triangle ABC, are tangent to the lines ℓ, AB, and AC as shown in the diagram. The lines DE and FG intersect at point P, which lies on BC. Prove that P is the midpoint of BC.

(*Mykhailo Plotnikov*)



Solution. Let I be the incenter of triangle ABC, K and L the points of tangency of this circle with sides AB and AC, and O_1 and O_2 the centers of the two other circles from the problem statement (Fig. 5). Since $AD = AE$ as tangents drawn from a single point to a circle, $\angle ADE = \angle AED$. But $\angle ADE = \angle BPE$ (corresponding angles with parallel lines) and $\angle AED = \angle BEP$ (vertical angles). Hence $\angle BEP = \angle BPE$, triangle BPE is isosceles, and $BP = BE$. Similarly, $CP = CG$, so it suffices to establish that $BE = CG$.

The right triangles O_1AE and IBK are congruent by a leg and an acute angle ($O_1E = IK$ as radii of congruent circles, $\angle O_1AE = \frac{1}{2}\angle DAE = \frac{1}{2}\angle PBE = \angle IBK$). Thus, $AE = BK$, and consequently $BE = AB - AE = AB - BK = AK$. Similarly, $CG = AL$, and noting that $AK = AL$ as tangents from a single point, the proof is complete.

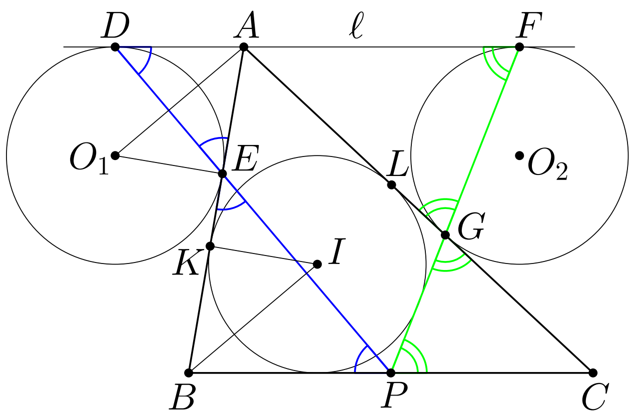


Fig. 5

6. In an isosceles triangle ABC with $\angle BAC = 108^\circ$, the bisector of angle ABC intersects the circumcircle of the triangle at point D. Point E on segment BC is such that $AB = BE$. Prove that the perpendicular bisector of CD is tangent to the circumcircle of triangle ABE.

(*Bohdan Zheliabovskyi*)

Solution. The base angles of isosceles triangle ABC are $\angle ABC = \angle BCA = 36^\circ$. Let O be the circumcenter of triangle ABC and ℓ the perpendicular bisector of CD (Fig. 6). From the isosceles triangle ABE, we find $\angle AEB = 90^\circ - \frac{1}{2}\angle ABC = 72^\circ$. Also, $\angle AOB = 2\angle ACB = 72^\circ$, since the central angle is twice the inscribed angle. Hence, the circumcircle of triangle ABE passes through point O. The line ℓ also passes through point O. We will show that ℓ is tangent to the circumcircle of triangle ABE at this point.

Since $\angle ACD = \angle ABD = \frac{1}{2}\angle ABC = 18^\circ$ and $\angle BCD = \angle BCA + \angle ACD = 54^\circ$, we have $\angle ABC + \angle BCD = 90^\circ$. Thus, $AB \perp CD$, and therefore $\ell \parallel AB$. But triangle AOB is isosceles, so the tangent to its circumcircle at point O is parallel to AB, implying that this tangent is the line ℓ.

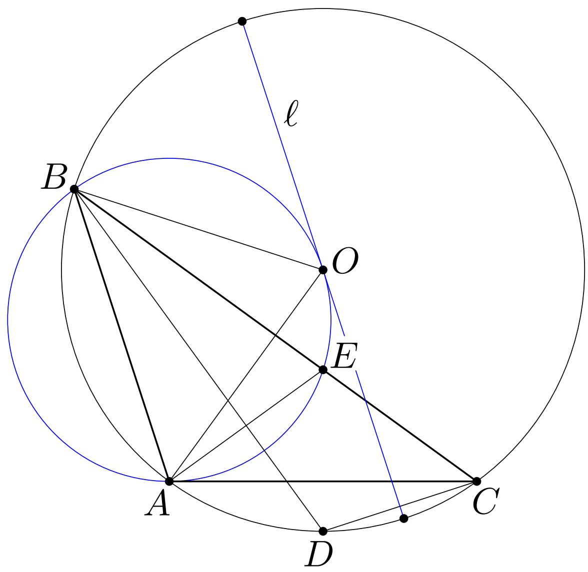


Fig. 6

9th Grade

1. In an acute triangle ABC, the altitudes BD and CE intersect at point H. A point F is chosen on side AC, such that $FH \perp CE$. Segment FE intersects the circumcircle of triangle CDE at point K. Prove that $HK \perp EF$.

(*Matthew Kurskyi*)

Solution.

Points D and E lie on the circle with diameter BC. By the problem statement, point K also belongs to this circle, so $\angle KDB = 180^\circ - \angle KEB = \angle KEA$ (Fig. 1). Since $FH \perp CE$ and $AB \perp CE$, it follows that $FH \parallel AB$. Hence, $\angle HFE = \angle KEA$. Therefore, $\angle HFE = \angle KDB$, and quadrilateral $DFKH$ is cyclic. Thus, $\angle FKH = 180^\circ - \angle HDF = 90^\circ$, which means $HK \perp EF$.

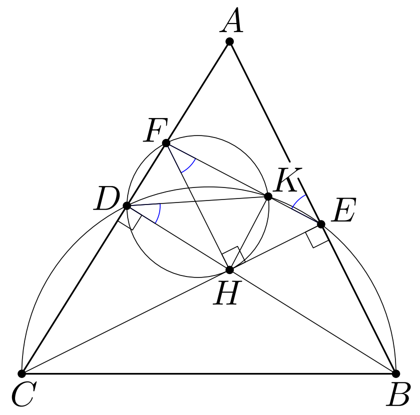


Fig. 1

2. Let BC and BD be the tangents drawn from point B to the circle with diameter AC, and let E be the second intersection point of line CD with the circumcircle of triangle ABC. Prove that $CD = 2DE$.

(*Matthew Kurskyi*)

Solution.

Let O be the midpoint of AC and F the intersection point of BO with CD (Fig. 2). The right triangles OBC and OBD are congruent by three sides ($OC = OD$ as radii, $BC = BD$ as tangents), so BF is the angle bisector, altitude, and median of isosceles triangle CBD. The right triangles ACB and EFB are similar, since $\angle FEB = \angle CEB = \angle CAB$. Since BO is the median of triangle ACB and $\angle OBC = \angle DBF$, it follows that BD is the median of triangle EFB. Hence, $CF = FD = DE$, and thus $CD = 2DE$.

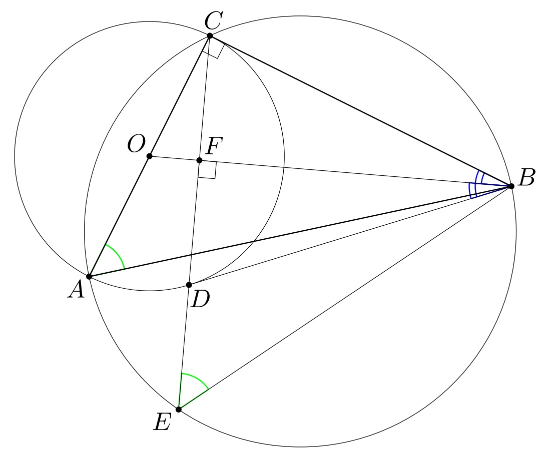


Fig. 2

3. Given a triangle ABC, with a marked point I as its incenter, and K_1 and K_2 being the points of tangency of the incircle with sides BC and AC, respectively.

Using a compass and a ruler, construct the center of the excircle of triangle CK_1K_2 that is tangent to CK_2, using at most 4 lines (a line is a straight line or a circle).

(*Hryhorii Filippovskyi and Volodymyr Brayman*)

Solution.

First, draw two lines: the incircle of triangle ABC (with center I and radius IK_1) and the line CI. Let these intersect at points D and E, where $CD < CE$ (Fig. 3). We will show that D is the center of the incircle of triangle CK_1K_2. Indeed, denote the center of the incircle by D'. Point D' lies on CI, and since I is the midpoint of the arc K_1K_2 of the circumcircle of triangle CK_1K_2, by the “Incenter lemma”, we have $ID' = IK_2 = IK_3$. Thus, points D and D' coincide. Since $\angle DK_2E = 90°$, K_2E is the angle bisector of the exterior angle at vertex K_2 of triangle CK_1K_2. Next, draw two more lines: EK_2 and K_1D. Their intersection gives the required center of the excircle.

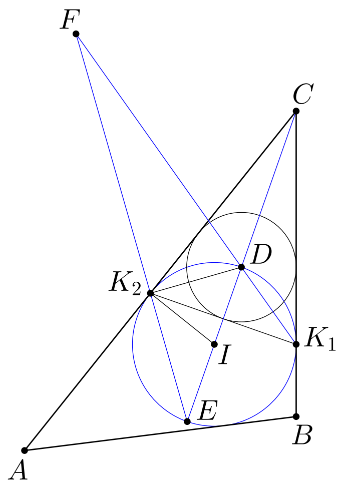


Fig. 3

Note. Point E is the center of the excircle of triangle CK_1K_2 that is tangent to K_1K_2.

4. Let BE and CF be the altitudes of an acute triangle ABC, H its orthocenter, M the midpoint of BC, K and L the intersection points of the perpendicular bisector of BC with BD and CE, respectively, and Q the orthocenter of triangle KLH. Prove that Q lies on the median AM.

(*Bohdan Zheliabovskyi*)

Solution.

Triangles ABC and HLK are similar because the corresponding sides of these triangles are perpendicular, and therefore their corresponding angles are equal. Let AD and HP be the altitudes of these triangles, drawn to BC and KL, respectively, and let Q' be the intersection point of AM with HP (Fig. 4). To show that Q and Q' coincide, it suffices to prove that $HQ' : Q'P = AH : HD$. Since the right triangles $AQ'H$ and $MQ'P$ are similar, $HQ' : Q'P = AH : MP$. It remains to observe that $MP = HD$, because $MPHD$ is a rectangle.

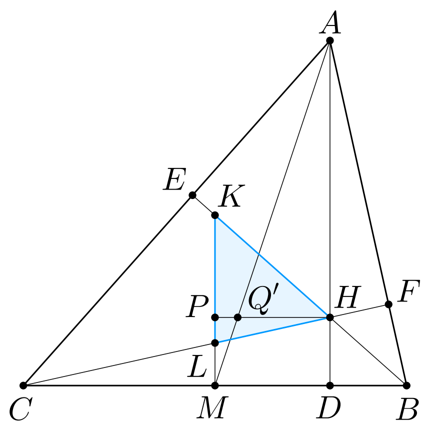


Fig. 4

5. Let I be the incenter of triangle ABC, and K the point of tangency of the incircle with side BC. Points X and Y are chosen on segments BI and CI, respectively, such that $KX \perp AB$ and $KY \perp AC$. The circumcircle of triangle XYK meets BC again at point D (other than point K). Prove that $AD \perp BC$.

(*Matthew Kurskyi*)

Solution. Let KX and KY intersect AB and AC at points E and F, respectively, and let AH be the altitude of triangle ABC (Fig. 5). We will show that points X, Y, K, H lie on the same circle. This will imply that points D and H coincide. The right triangles BEX and BKI are similar, so $\frac{BX}{BI} = \frac{BE}{BK} = \cos B = \frac{BH}{BA}$. Hence, $BX : BH = BI : BA$, which means triangles BHX and BAI are similar by two sides and the included angle. Thus, $\angle BHX = \angle BAI = \frac{A}{2}$. Similarly, $\angle CHY = \frac{A}{2}$, so $\angle XHY = 180^\circ - \angle BHX - \angle CHY = 180^\circ - A$. From quadrilateral $AEKF$, we find $\angle XKY = \angle EKF = 180^\circ - A = \angle XHY$, so points X, Y, K, H lie on the same circle, completing the proof.

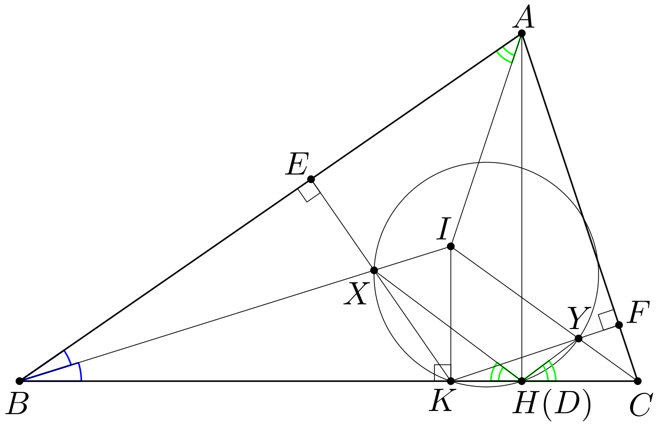
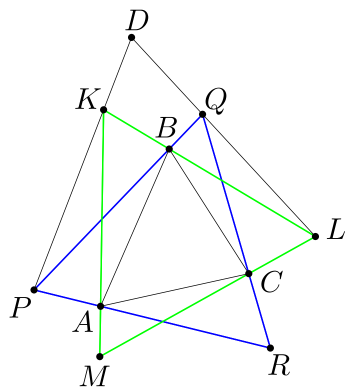


Fig. 5

6. Around an acute triangle ABC, equilateral triangles KLM and PQR are constructed as shown in the diagram. Lines PK and QL intersect at point D. Prove that $\angle ABC + \angle PDQ = 120^\circ$.

(*Yurii Biletskyi*)



Solution. Since $\angle APB = \angle AKB = 60^\circ$, quadrilateral $APKB$ is cyclic, and similarly, quadrilateral $BQLC$ is cyclic. Let the circles circumscribed around these quadrilaterals intersect at points B and O (Fig. 6). Then $\angle AOB = 120^\circ$. We will show that quadrilateral $PDQO$ is cyclic. Indeed, let $\angle POA = \angle PKA = \alpha$ and $\angle BOQ = \angle BLQ = \beta$. Then

$$\angle POQ = \angle AOB - \angle POA + \angle BOQ = 120^\circ - \alpha + \beta,$$

$$\angle PDQ = \angle PKL - \angle KLQ = 60^\circ + \alpha - \beta,$$

hence $\angle POQ + \angle PDQ = 180^\circ$. Now,

$$\begin{aligned}\angle ABC + \angle PDQ =& \angle ABO + \angle OBC + \angle PDO + \angle ODQ = \\ =& \angle APO + \angle OQC + \angle PQO + \angle OPQ = \\ =& \angle APQ + \angle PQC = 60^\circ + 60^\circ = 120^\circ.\end{aligned}$$

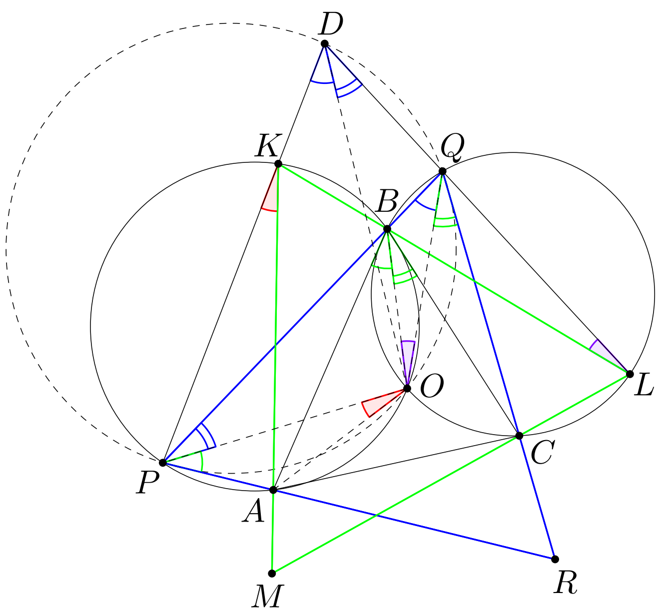


Fig. 6

10-11th Grade

1. Circles ω_1 and ω_2 are tangent to a line ℓ at points A and B, respectively, and are tangent to each other externally at point D. A point E is chosen arbitrarily on the minor arc BD of circle ω_2. The line DE meets circle ω_1 at point C for the second time. Prove that $BE \perp AC$.

(*Yurii Biletskyi*)

Solution. Let O_1 and O_2 be the centers of circles ω_1 and ω_2, respectively, and let F be the intersection point of lines AC and BE (Fig. 1). We will show that points A, B, D, and F lie on the same circle. Indeed, in the isosceles triangles O_1CD and O_2DE, we have $\angle O_1DC = \angle O_2DE$ as vertical angles, thus $\angle CO_1D = \angle DO_2E$. Therefore,

$$\angle CAD = \tfrac{1}{2}\angle CO_1D = \tfrac{1}{2}\angle DO_2E = \angle DBE,$$

which implies $\angle FAD = \angle FBD$, and quadrilateral $ABDF$ is cyclic.

Since $AO_1 \parallel BO_2$, it follows that $\angle AO_1D + \angle BO_2D = 180^\circ$. From the isosceles triangles AO_1D and BO_2D, we find

$$\angle O_1DA + \angle O_2DB = (90^\circ - \tfrac{1}{2}\angle AO_1D) + (90^\circ - \tfrac{1}{2}\angle BO_2D) = 90^\circ.$$

Hence, $\angle ADB = 90^\circ$, and therefore $\angle AFB = 90^\circ$.

Solution 2. Let the homothety centered at D, which maps circle ω_2 onto circle ω_1, map radius O_2B to radius O_1G (Fig. 2). This homothety maps triangle BED to triangle GCD, so $GC \parallel BE$. Since $O_2B \parallel O_1A$ and $O_2B \parallel O_1G$, we have $G - O_1 - A$ as a diameter of circle ω_1. Thus, $AC \perp GC$, which implies $AC \perp BE$.

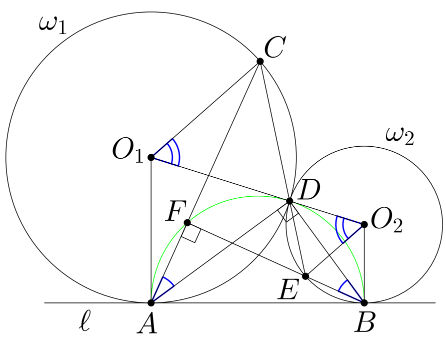


Fig. 1

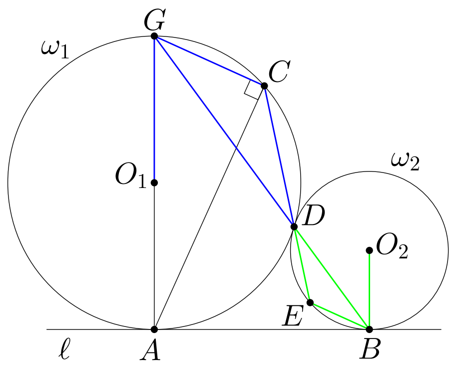


Fig. 2

2. Let I be the incenter of triangle ABC, where $\angle A = 60^\circ$, and let D be the point of tangency of the incircle with side BC. Points X and Y are chosen on segments BI and CI, respectively, such that $DX \perp AB$ and $DY \perp AC$. A point

Z is chosen such that triangle XYZ is equilateral, and points Z and I lie on the same side of line XY. Prove that $AZ \perp BC$.

(*Matthew Kurskyi*)

Solution. Let the incircle of triangle ABC touch sides AC and AB at points E and F, respectively (Fig. 3). Since $CI \perp ED$, point Y is the orthocenter of triangle DEC. Thus, $EY \perp BC$ and $ID \perp BC$, which implies $EY \parallel ID$. Similarly, $EI \parallel YD$, so $EIDY$ is a parallelogram, and $\overrightarrow{EY} = \overrightarrow{ID}$. Analogously, $\overrightarrow{FX} = \overrightarrow{ID}$.

The equilateral triangles AEF and ZYX are similar and equally oriented, and under the parallel translation by vector $\overrightarrow{EY} = \overrightarrow{FX}$, points E and F map to points Y and X, respectively. Thus, point A maps to point Z, $\overrightarrow{AZ} = \overrightarrow{EY}$, and therefore $AZ \perp BC$.

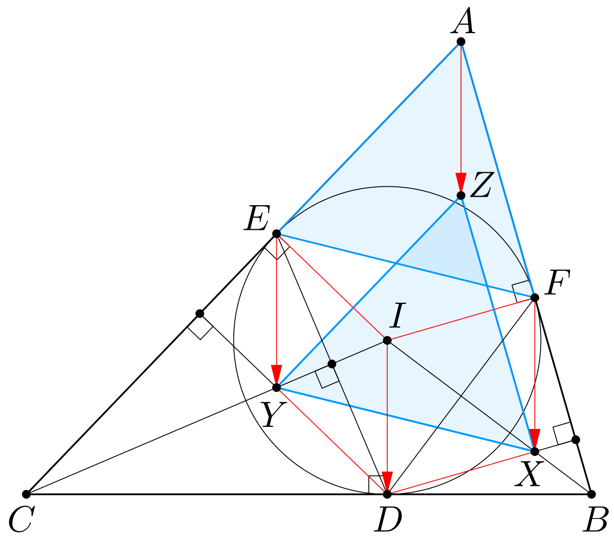


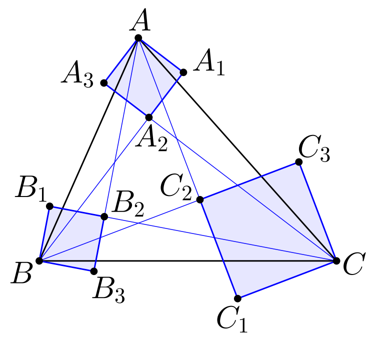
Fig. 3

3. Given an acute triangle ABC. Squares $AA_1A_2A_3$, $BB_1B_2B_3$, and $CC_1C_2C_3$ are positioned such that the lines A_1A_2, B_1B_2, and C_1C_2 pass through points B, C, and A, respectively, and the lines A_2A_3, B_2B_3, and C_2C_3 pass through points C, A, and B, respectively. Prove that

a) the lines AA_2, B_1B_3, and C_1C_3 are concurrent;

b) the lines AA_2, BB_2, and CC_2 are concurrent.

(*Mykhailo Plotnikov*)



Solution. a) The lines B_1B_3, AA_2, and C_1C_3 contain the bisectors of the right angles $\angle BB_1C$, $\angle BA_2C$, and $\angle BC_3C$, so all of them pass through point A', the midpoint of the semicircle constructed on BC as a diameter outside triangle ABC (Fig. 4).

b) Let A', B', and C' be the midpoints of the semicircles with diameters BC, AC, and AB, constructed outside triangle ABC. From part a), line A_1A_3 passes through points B' and C', and line AA_2 passes through point A'. Since $A_1A_3 \perp AA_2$ as the diagonals of a square, line AA_2 contains the altitude of triangle $A'B'C'$. The lines BB_2 and CC_2 also contain the altitudes of triangle $A'B'C'$, so the lines AA_2, BB_2, and CC_2 meet at the orthocenter of this triangle.

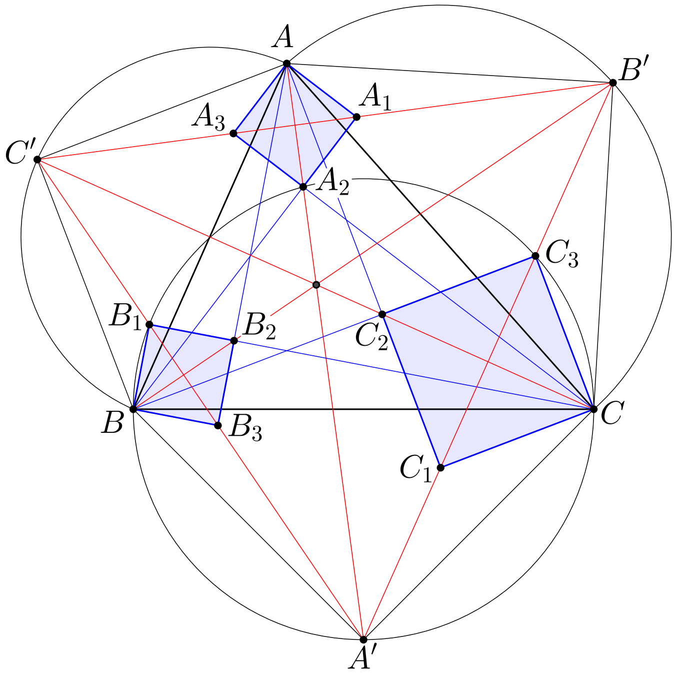


Fig. 4

4. On a semicircle with diameter AB, a point C is chosen arbitrarily. Let P and Q be points on segment AB such that $AP = AC$ and $BQ = BC$, and let O and H be the circumcenter and orthocenter of triangle CPQ, respectively. Prove that for all possible positions of point C, line OH passes through a fixed point.

(*Mykhailo Sydorenko*)

Solution. We will show that line OH always passes through point N, the midpoint of the semicircle with diameter AB (Fig. 5).

First, we prove that $\angle PCQ = 45°$. Indeed, from the isosceles triangles ACP and BCQ, we obtain that $\angle QPC = 90° - \frac{1}{2}\angle CAB$ and $\angle PQC = 90° - \frac{1}{2}\angle CBA$, so

$$\begin{aligned}\angle PCQ &= 180° - \angle QPC - \angle PQC = \\ &= 180° - (90° - \tfrac{1}{2}\angle CAB) - (90° - \tfrac{1}{2}\angle CBA) = \\ &= \tfrac{1}{2}(\angle CAB + \angle CBA) = 45°.\end{aligned}$$

Hence, $\angle POQ = 2\angle PCQ = 90°$, and POQ is an isosceles right triangle. But ANB

is also an isosceles right triangle, so triangles ANB and QOP are homothetic.

Since triangles ACP and BCQ are isosceles, $AO \perp CP$ and $BO \perp CQ$, implying $AO \parallel QH$ and $BO \parallel PH$. Thus, triangles AOB and QHP are also homothetic. Consequently, there exists a homothety that maps triangles ANB and AOB onto triangles QOP and QHP. This homothety maps segment NO onto segment OH, and since these segments share a common point, they lie on the same line. Thus, line OH always passes through point N.

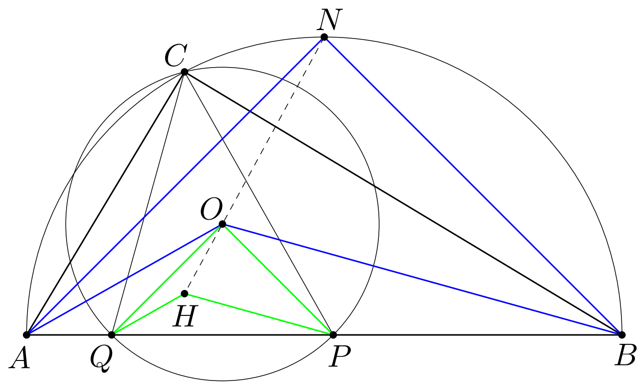


Fig. 5

5. Given a scalene triangle ABC, with the incenter I marked, and the points of tangency of the incircle with sides BC, AC, and AB marked as K_1, K_2, and K_3, respectively. Using only a ruler, construct the circumcenter of triangle ABC.

(*Hryhorii Filippovskyi*)

Solution. The construction consists of two steps.

Step 1. Find the midpoints of the sides of triangle M_1, M_2, and M_3.

Method I. Let D be the intersection point of lines K_1K_3 and CI. We will show that D lies on line M_2M_3 (Fig. 6). Since $\angle K_1DC = \angle K_3K_1B - \angle ICB = \frac{A+C}{2} - \frac{C}{2} = \frac{A}{2} = \angle K_3AI$, points A, I, D, K_3 lie on the same circle. Hence, $\angle IDA = \angle IK_3A = 90^\circ$. Let line AD intersect BC at point F. Then CD is the angle bisector and altitude of triangle ACF, so this triangle is isosceles. Hence, D is the midpoint of AF, and therefore lies on line M_2M_3. Now let E be the intersection point of lines K_1K_2 and BI. This point also lies on M_2M_3, so line DE intersects AB and AC at points M_3 and M_2. Similarly, construct line M_1M_2 to find point M_1.

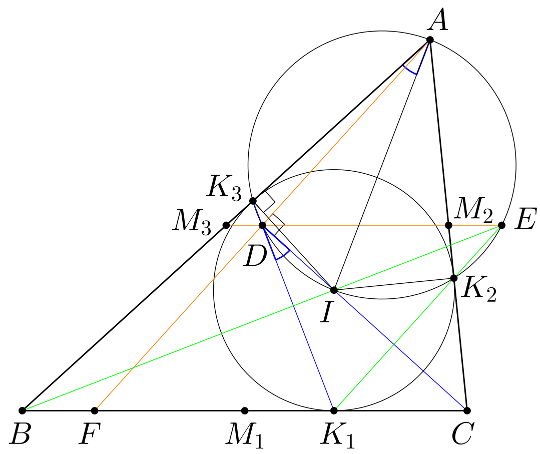


Fig. 6

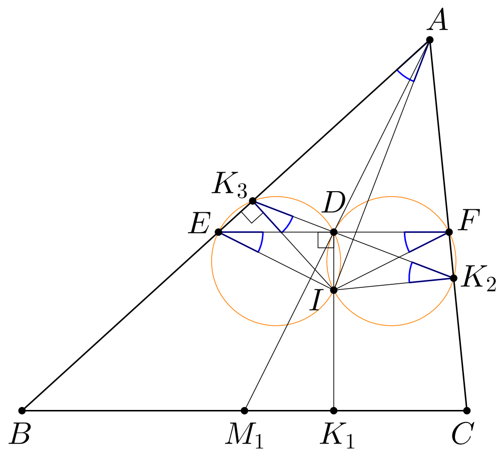


Fig. 7

Method II. Let D be the intersection point of lines K_1I and K_2K_3 (Fig. 7). We will show that line AD passes through point M_1. To do this, draw segment $EF \parallel BC$ through point D ($E \in AB$, $F \in AC$) and show that D is the midpoint of EF. Indeed, quadrilateral AK_2IK_3 is inscribed in a circle with diameter AI, so $\angle IK_3K_2 = \angle IK_2K_3 = \frac{A}{2}$. Since $\angle EK_3I = \angle EDI = 90^\circ$, points E, K_3, D, I lie on the same circle, and similarly points I, D, F, K_2 lie on the same circle. Thus, $\angle IED = \angle IK_3D = \frac{A}{2} = \angle IK_2D = \angle IFD$. Hence, triangle IEF is isosceles, and its altitude ID is the median. Similarly, construct points M_2 and M_3.

Step 2. Construct the perpendicular bisectors of the sides of the triangle. Let G be the intersection point of AM_1 and M_2M_3, and P the intersection point of M_1M_3 and CG (Fig. 8). Then segments M_3G and GM_2 are the midlines of triangles M_1PC and BAM_1, so $PA \parallel BC$. Let Q be the intersection point of K_1I and PA, S the intersection point of M_1Q and M_2M_3, and T the intersection point of K_1S and PA. It is easy to verify that M_1K_1QT is a rectangle, so $M_1T \perp BC$. Similarly, construct the perpendicular bisector of another side to find the circumcenter of triangle ABC.

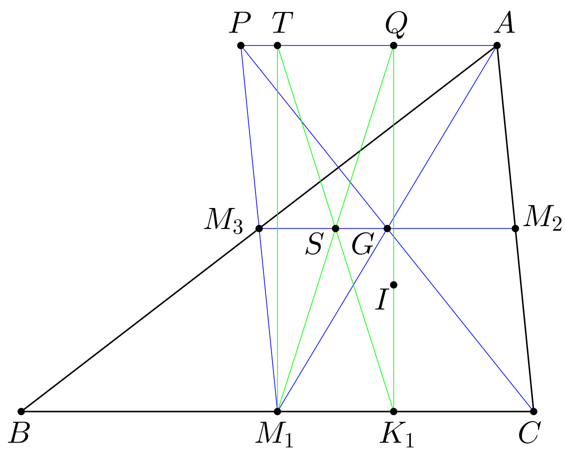


Fig. 8

6. Given a scalene triangle ABC. Through point B, a line ℓ is drawn that does not intersect the triangle and forms distinct angles with sides AB and BC. Let M be the midpoint of AC, and let H_a and H_c be the feet of the perpendiculars drawn from points A and C to ℓ. The circumcircle of triangle MBH_a intersects AB at point A_1, and the circumcircle of triangle MBH_c intersects BC at point C_1. Point A_2 is symmetric to A with respect to point A_1, and point C_2 is symmetric to C with respect to point C_1. Prove that the lines ℓ, AC_2, and CA_2 are concurrent.

(*Yana Kolodach*)

Solution. Extend AH_a such that $H_aN_a = AH_a$ and CH_c such that $H_cN_c = CH_c$ (Fig. 9). Observe that lines BC and BN_c are symmetric with respect to ℓ, so points A, B, and N_c are not collinear. Let the circumcircle of triangle ABN_c intersect ℓ again at point E.

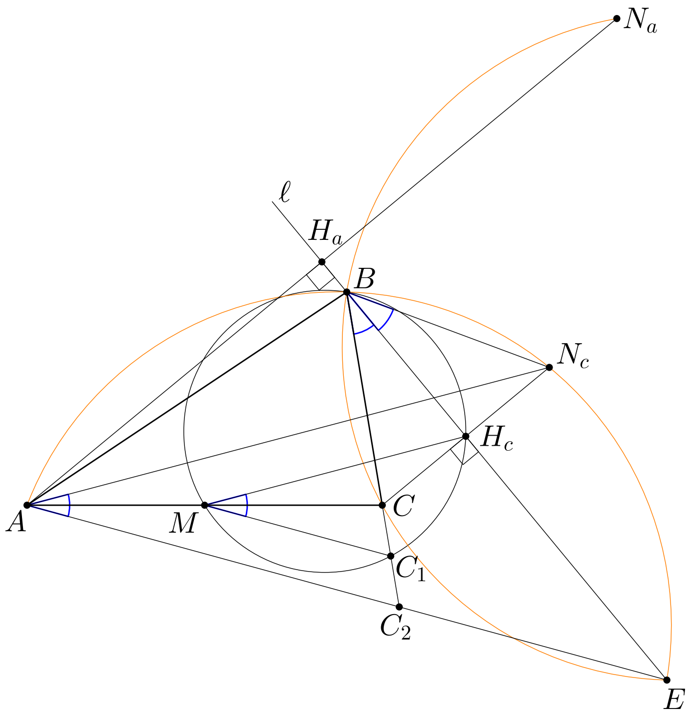


Fig. 9

We will show that line AC_2 passes through point E. Indeed[1],

$$\angle N_cAE = \angle N_cBE = \angle N_cBH_c = \angle H_cBC_1 = \angle H_cMC_1.$$

Since MH_c and MC_1 are the midlines of triangles ACN_c and ACC_2, respectively, we have

$$\angle N_cAC_2 = \angle H_cMC_1 = \angle N_cAE,$$

so line AC_2 passes through point E. Similarly, line A_2C passes through the intersection point of the circumcircle of triangle N_aBC with line ℓ, distinct from B. But triangles ABN_c and N_aBC are symmetric with respect to line ℓ. Hence, the circumcircle of triangle N_aBC also intersects line ℓ at point E, completing the proof.

[1]This reasoning corresponds to the configuration depicted in Fig. 9; in other cases, the arguments will be analogous.