8th Grade

1. Let BE and CF be the medians of an acute triangle ABC. On the line BC, points $K \neq B$ and $L \neq C$ are chosen such that $BE = EK$ and $CF = FL$. Prove that $AK = AL$.

(*Heorhii Zhilinskyi*)

Solution 1. Let AA_1, EE_1, and FF_1 be the altitudes of triangles ABC, BEK, and CFL, respectively (Fig. 1). Since $EE_1 \parallel AA_1 \parallel FF_1$, it follows that EE_1 and FF_1 are the midlines of triangles AA_1C and AA_1B. Denote $BF_1 = F_1A_1 = x$ and $A_1E_1 = E_1B = y$. Since triangles BEK and CFL are isosceles, E_1 and F_1 are midpoints of BK and CL, respectively. Therefore, $E_1K = BE_1 = 2x + y$, $LF_1 = F_1C = x + 2y$, and hence $LA_1 = A_1K = 2x + 2y$. Thus, in triangle KAL, the height AA_1 is also the median, which implies $AK = AL$.

Solution 2. Extend BE to a point N such that $EN = BE$, and extend CF to a point M such that $FM = CF$ (Fig. 2). Then $ABCN$ and $ACBM$ are parallelograms, so $MA = BC = AN$, $MA \parallel BC$, and $AN \parallel BC$. Hence, $MN \parallel BC$ and A is the midpoint of MN. In triangle BNK, the median KE equals half of the side BN, so this triangle is right-angled. Thus, $NK \perp BC$, and similarly, $ML \perp BC$. It follows that $KLMN$ is a rectangle. Since A is the midpoint of MN, the right triangles ANK and AML are congruent by two legs, and therefore $AK = AL$.

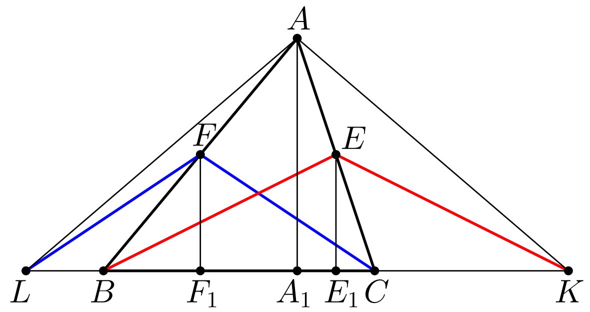

Fig. 1.

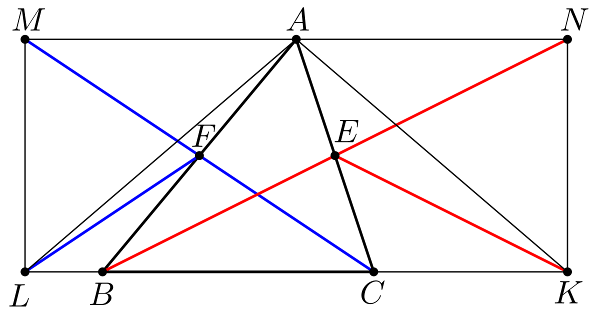

Fig. 2.

2. Let I be the incenter and O be the circumcenter of triangle ABC, where $\angle A < \angle B < \angle C$. Points P and Q are such that $AIOP$ and $BIOQ$ are isosceles trapezoids ($AI \parallel OP$, $BI \parallel OQ$). Prove that $CP = CQ$.

(*Volodymyr Brayman and Matthew Kurskyi*)

Solution. The diagonals of an isosceles trapezoid are equal, so $IP = AO = BO = IQ$ (Fig. 3). We will prove that $\angle CIP = \angle CIQ$. From this, it follows that triangles CIP and CIQ are congruent by SAS theorem, which implies $CP = CQ$.

Let $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$, where $\alpha < \beta < \gamma$. In the isosceles triangle AOC, the angle at the vertex is 2β, and the base angle is

$$\angle CAO = 90^\circ - \beta > 90^\circ - \frac{1}{2}(\beta + \gamma) = \frac{\alpha}{2} = \angle CAI.$$

Similarly, $\angle CBO = 90^\circ - \alpha > \frac{\beta}{2} = \angle BCI$. Therefore, point O lies inside angle AIB, and points P and Q lie inside angles AIO and BIO, respectively. Consequently,

$$\angle CIP = \angle CIA + \angle AIP \quad \text{and} \quad \angle CIQ = \angle CIB + \angle BIQ.$$

Since $\angle CIA = 90^\circ + \frac{\beta}{2}$ and from the isosceles trapezoid $\angle AIP = \angle OAI = \angle CAO - \angle CAI = 90^\circ - \beta - \frac{\alpha}{2}$, we find that

$$\angle CIP = 90^\circ + \frac{\beta}{2} + 90^\circ - \beta - \frac{\alpha}{2} = 180^\circ - \frac{\beta}{2} - \frac{\alpha}{2} = 90^\circ + \frac{\gamma}{2}.$$

Similarly, $\angle CIQ = 90^\circ + \frac{\gamma}{2}$, which completes the proof.

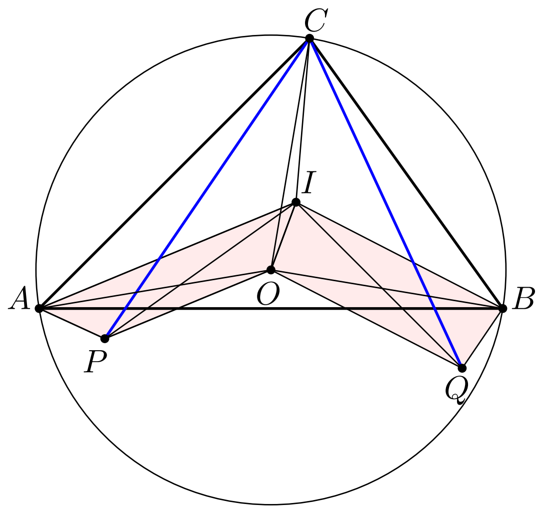

Fig. 3.

3. Let W be the midpoint of the arc BC of the circumcircle of triangle ABC, such that W and A lie on opposite sides of line BC. On sides AB and AC, points P and Q are chosen respectively so that $APWQ$ is a parallelogram, and on side BC, points K and L are chosen such that $BK = KW$ and $CL = LW$. Prove that the lines AW, KQ, and LP are concurrent.

(*Matthew Kurskyi*)

Solution. Let $\angle BAC = 2\alpha$. Since triangle BKW is isosceles (Fig. 4), we have

$$\angle BWK = \angle WBC = \angle WAC = \alpha.$$

Thus,

$$\angle WKC = \angle BWK + \angle WBC = 2\alpha.$$

Since $WQ \parallel AB$, it follows that

$$\angle WQC = 2\alpha = \angle WKC,$$

which means that quadrilateral $WKQC$ is cyclic. Similarly, quadrilateral $WBPL$ is cyclic. Therefore, $\angle WPL = \angle WCL = \alpha$ and $\angle KQW = \angle KCW = \alpha$, so $\angle BPL = \angle CQK = 3\alpha$.

Since the diagonal of the parallelogram $APWQ$ is the angle bisector of $\angle A$, the figure $APWQ$ is a rhombus. Let the lines PL and QK intersect AW at points D' and D'' respectively. Since $AP = AQ$, $\angle PAD' = \angle QAD'' = \alpha$, and $\angle APD' = \angle AQD'' = 180° - 3\alpha$, the triangles APD' and AQD'' are congruent. Therefore, $AD' = AD''$, which implies that the points D' and D'' coincide.

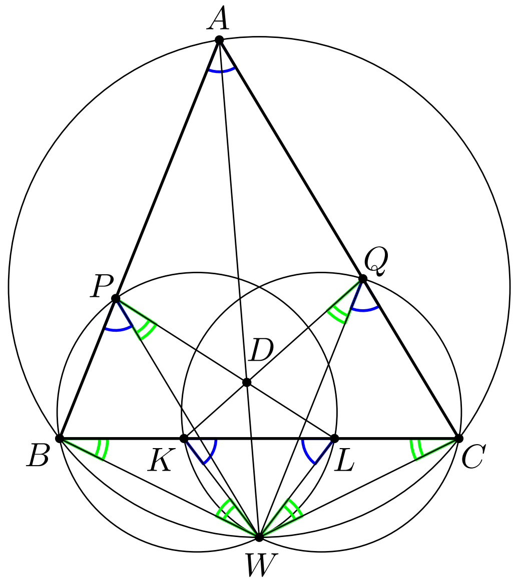

Fig. 4.

4. On side AB of an isosceles trapezoid $ABCD$ ($AD \parallel BC$), points E and F are chosen such that a circle can be inscribed in quadrilateral $CDEF$. Prove that the circumcircles of triangles ADE and BCF are tangent to each other.

(*Matthew Kurskyi*)

Solution. Let ω be the incircle of quadrilateral $CDEF$, and let ω_1 and ω_2 be the circumcircles of triangles ADE and BCF, respectively. Denote O, O_1, and O_2 as the centers of circles ω, ω_1, and ω_2, respectively, and let S be the intersection of lines AB and CD (Fig. 5). The point O lies on the angle bisector of $\angle ASD$, which is the perpendicular bisector of segments AD and BC. Therefore, the points O_1 and O_2 also lie on this bisector. We will show that the circles ω_1 and ω_2 pass through point O. Since the centers of these circles lie on the same line with O, it follows that O is the tangency point of ω_1 and ω_2.

Denote $\angle ASD = \alpha$. Then

$$\angle SAD = \angle SBC = 90° - \frac{\alpha}{2}.$$

Since circle ω is inscribed in triangle ESD, we have $\angle EOD = 90° + \frac{\alpha}{2}$. Therefore, $\angle EOD + \angle EAD = 180°$, which implies that point O lies on circle ω_1. Similarly, since circle ω is also the excircle for triangle FSC, we have $\angle FOC = 90^\circ - \frac{\alpha}{2}$. It follows that $\angle FOC + \angle FBC = 180^\circ$, which means that point O also lies on circle ω_2.

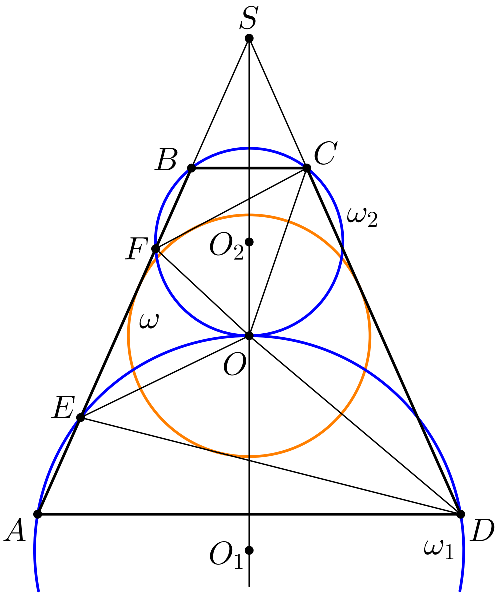

Fig. 5.

5. On side AC of triangle ABC, a point P is chosen such that $AP = \frac{1}{3}AC$, and on segment BP, a point S is chosen such that $CS \perp BP$. A point T is such that $BCST$ is a parallelogram. Prove that $AB = AT$.

(*Bohdan Zheliabovskyi*)

Solution. Extend BC beyond point B to a segment $BD = DC$, and extend AC beyond point A to a segment $AQ = AP$ (Fig. 6). Then $PQ = \frac{2}{3}AC = CP$, so BP is the midline of triangle CDQ. It follows that $DQ \parallel BP$. Since $BD = BC = ST$ and $BD \parallel ST$, quadrilateral $BSTD$ is a parallelogram. Therefore, $TD \parallel BP$, which implies that $D - T - Q$ are collinear and $BP \parallel TQ$.

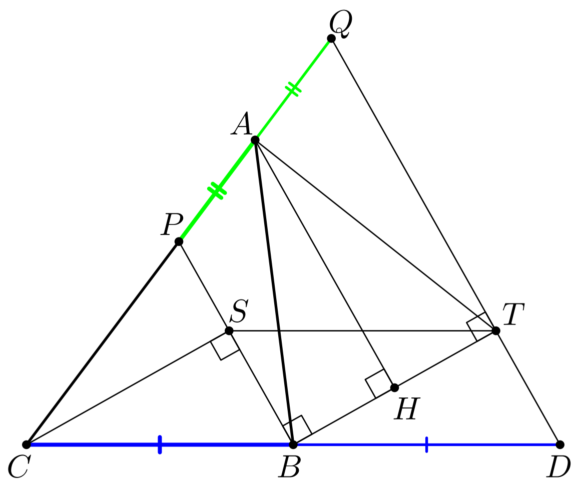

Fig. 6.

Since $BT \parallel CS$ and $CS \perp BP$, it follows that $PB \perp BT$. Thus, $PBTQ$ is a right trapezoid. Let AH be the altitude of triangle ABT. Then $AH \parallel BP$, and A is the midpoint of PQ. Hence, AH is the midline of the trapezoid $PBTQ$, so H is the midpoint of BT. Consequently, in triangle ABT, the altitude AH is also the median, which implies that $AB = AT$.

9th Grade

1. Inside triangle ABC, a point D is chosen such that $\angle ADB = \angle ADC$. The rays BD and CD intersect the circumcircle of triangle ABC at points E and F, respectively. On segment EF, points K and L are chosen such that $\angle AKD = 180^\circ - \angle ACB$ and $\angle ALD = 180^\circ - \angle ABC$, with segments EL and FK not intersecting line AD. Prove that $AK = AL$.

(*Matthew Kurskyi*)

Solution. Since $\angle AED = \angle ACB = 180^\circ - \angle AKD$, and points K and E lie on opposite sides of AD, quadrilateral $AKDE$ is cyclic. Similarly, quadrilateral $ALDF$ is also cyclic. Therefore,

$$\angle AKL = \angle ADE = 180^\circ - \angle ADB = 180^\circ - \angle ADC = \angle ADF = \angle ALK.$$

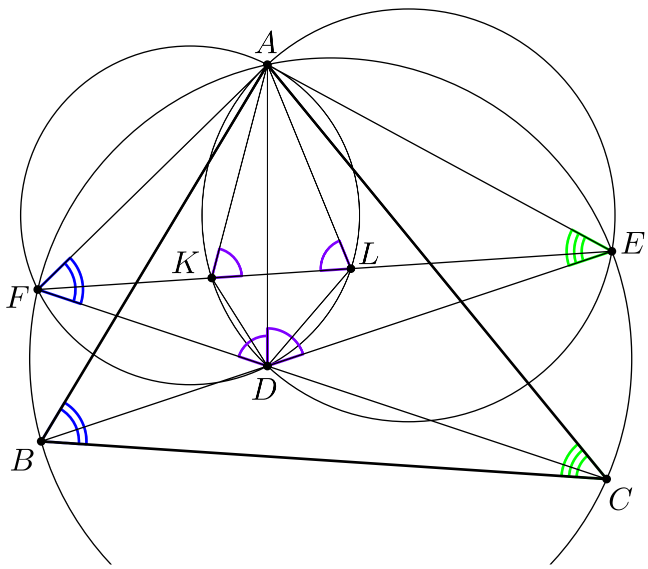

Fig. 1.

2. Let M be the midpoint of side BC of triangle ABC, and let D be an arbitrary point on the arc BC of the circumcircle that does not contain A. Let N be the midpoint of AD. A circle passing through points A, N, and tangent to AB intersects side AC at point E. Prove that points C, D, E, and M are concyclic.

(*Matthew Kurskyi*)

Solution. Since $\angle NAE = \angle DAC = \angle DBC$ and $\angle NEA = \angle BAD = \angle BCD$ (Fig. 2), triangles AEN and BCD are similar. Let K be the midpoint of AE. Since NK and DM are corresponding medians in similar triangles, we have $\angle NKE = \angle DMC$. Moreover, NK is the midline of triangle DAE, so $NK \parallel DE$. It follows that

$$\angle DEC = \angle NKE = \angle DMC,$$

and hence points C, D, E, and M are concyclic.

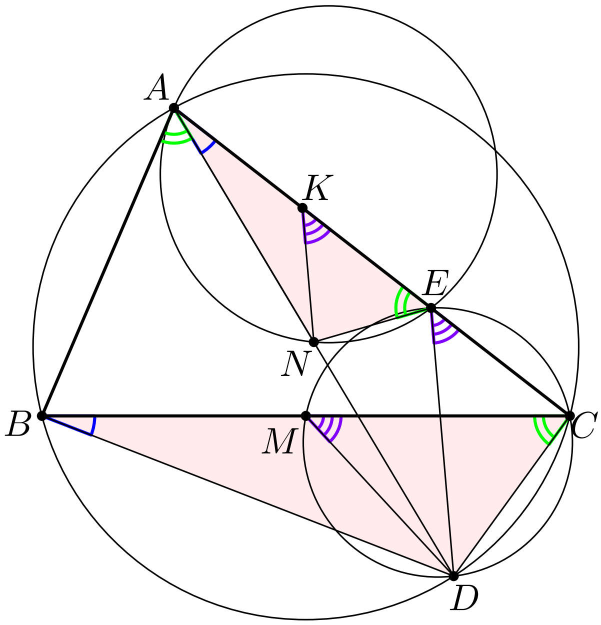

Fig. 2.

3. Let H be the orthocenter of an acute triangle ABC, and let AT be the diameter of the circumcircle of this triangle. Points X and Y are chosen on sides AC and AB, respectively, such that $TX = TY$ and $\angle XTY + \angle XAY = 90°$. Prove that $\angle XHY = 90°$.

(*Matthew Kurskyi*)

Solution. Since AT is the diameter, we have $\angle ABT = \angle ACT = 90°$. We will show that the right triangles XCT and TBY are congruent (Fig. 3). Indeed, $XT = TY$ by the condition, and since

$$\angle CTX + \angle BTY = \angle CTB - \angle XTY = 180° - \angle XAY - \angle XTY = 90°,$$

it follows that $\angle CXT = \angle BTY$. Therefore, $CX = BT$ and $BY = CT$.

Additionally, since $CH \perp AB$ and $TB \perp AB$, we have $TB \parallel CH$, and similarly, $TC \parallel BH$. Thus, $BHCT$ is a parallelogram, which implies $BH = CT = BY$ and $CH = BT = CX$.

Let $\angle BAC = \alpha$. Then $\angle BHC = 180° - \alpha$. Since $\angle ACH = \angle ABH = 90° - \alpha$, from the isosceles triangles BHY and CHX, we find

$$\angle BHY = \angle CHX = 45° + \frac{\alpha}{2}.$$

Thus,

$$\angle XHY = 360° - \angle BHC - \angle BHY - \angle CHX = 360° - (180° - \alpha) - 2\left(45° + \tfrac{\alpha}{2}\right) = 90°.$$

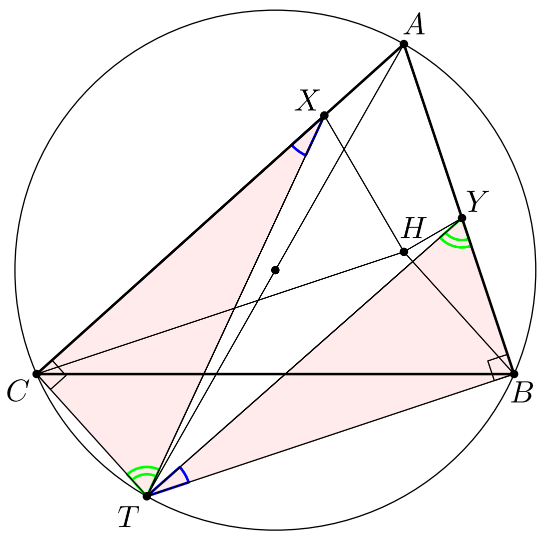

Fig. 3.

4. Let ω be the circumcircle of triangle ABC, where $AB > AC$. Let N be the midpoint of arc $\smile BAC$, and D a point on the circle ω such that $ND \perp AB$. Let I be the incenter of triangle ABC. Reconstruct triangle ABC, given the marked points A, D, and I.

(*Oleksii Karlyuchenko and Hryhorii Filippovskyi*)

Solution. Let NW be the diameter of circle ω (Fig. 4). Since $DW \perp ND$ and $AB \perp ND$, we have $DW \parallel AB$. Therefore, $\smile AD = \smile BW$, which implies $AD = BW$. By the incenter–excenter lemma, $IW = BW = CW$. Thus the triangle can be reconstructed as follows:

1) extend AI beyond point I by segment $IW = AD$ to obtain point W;

2) construct the circumcircle ω as the circle circumscribed around triangle ADW;

3) the circle centered at W with radius WI intersects ω at points B and C.

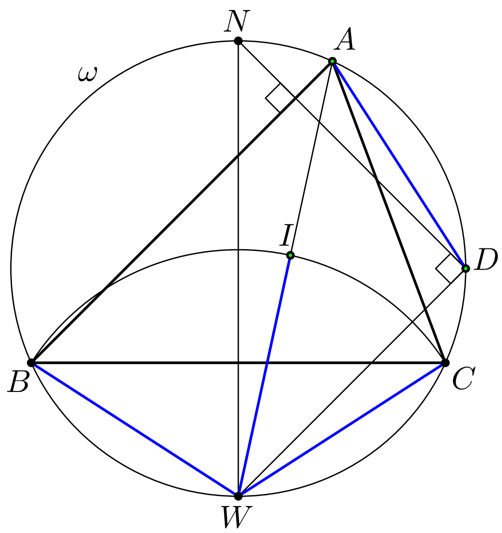

Fig. 4.

5. Let AL be the bisector of triangle ABC, O the center of its circumcircle, and D and E the midpoints of BL and CL, respectively. Points P and Q are chosen on segments AD and AE such that $APLQ$ is a parallelogram. Prove that $PQ \perp AO$.

(*Mykhailo Plotnikov*)

Solution. If $AB = AC$, the statement is obvious. Henceforth, assume without loss of generality that $\frac{AB}{AC} = t > 1$.

Let T be the intersection of the diagonals of the parallelogram $APLQ$ (Fig. 5). We will prove that PQ is tangent to the circumcircle of triangle TDE. Since triangles TDE and ABC are homothetic, it follows that line PQ is parallel to the tangent to the circumcircle of triangle ABC at A, and thus perpendicular to the radius AO.

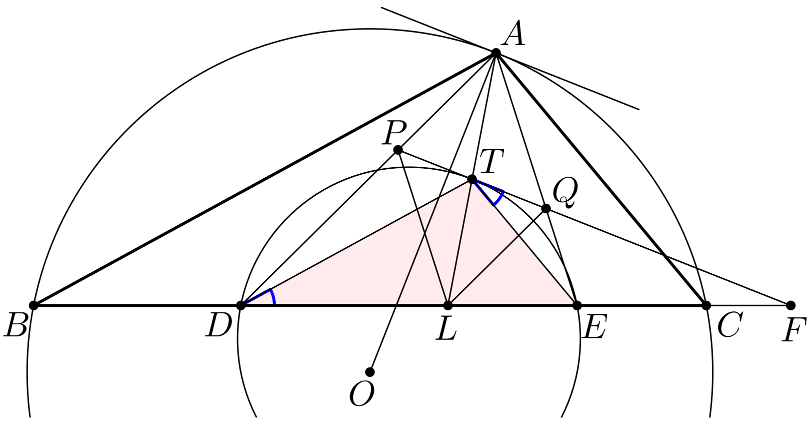

Fig. 5.

Let line PQ intersect BC at point F, and let the tangent to the circumcircle of triangle TDE at T intersect BC at point F'. We will show that $F' = F$. Both points F and F' lie outside segment BC, so it suffices to prove that $\frac{DF}{EF} = \frac{DF'}{EF'}$.

By the property of the bisector,

$$\frac{DL}{EL} = \frac{BL}{CL} = \frac{AB}{AC} = t.$$

Since $PL \parallel AE$ and $QL \parallel AD$, by the theorem on proportional segments,

$$\frac{DP}{PA} = \frac{AQ}{QE} = \frac{DL}{EL} = t.$$

By Menelaus' theorem,

$$\frac{DP}{PA} \cdot \frac{AQ}{QE} \cdot \frac{EF}{DF} = 1,$$

which implies that

$$\frac{DF}{EF} = \frac{DP}{PA} \cdot \frac{AQ}{QE} = t^2.$$

Since $\angle ETF' = \angle F'DT$, triangles ETF' and $F'DT$ are similar. Therefore,

$$\frac{TF'}{EF'} = \frac{DF'}{TF'} = \frac{DT}{ET} = \frac{AB}{AC} = t,$$

and so

$$\frac{DF'}{EF'} = \frac{DF'}{TF'} \cdot \frac{TF'}{EF'} = t^2 = \frac{DF}{EF}.$$

This completes the proof.

10–11th Grade

1. Let I and O be the incenter and circumcenter of the right triangle ABC ($\angle C = 90°$), and let K be the tangency point of the incircle with AC. Let P and Q be the points where the circumcircle of triangle AOK intersects OC and the circumcircle of triangle ABC, respectively. Prove that points C, I, P, and Q are concyclic.

(*Mykhailo Sydorenko*)

Solution. Since CIK is a right triangle with a $45°$ angle, we have $IK = CK$. The quadrilateral $AKPO$ is cyclic (Fig. 1), so

$$\angle KPC = 180° - \angle KPO = \angle KAO = \angle ACO.$$

Therefore, triangle KPC is isosceles, which implies $PK = CK$. Thus, K is the center of the circumcircle of triangle CIP. It remains to prove that point Q also lies on this circle, i.e., $QK = PK$. Since $\angle QAP = \angle QOP = \angle QOC = 2\angle QAC$, the line AC is the bisector of $\angle QAP$. Hence, K is the midpoint of the arc $\smile QKP$, and therefore $QK = PK$.

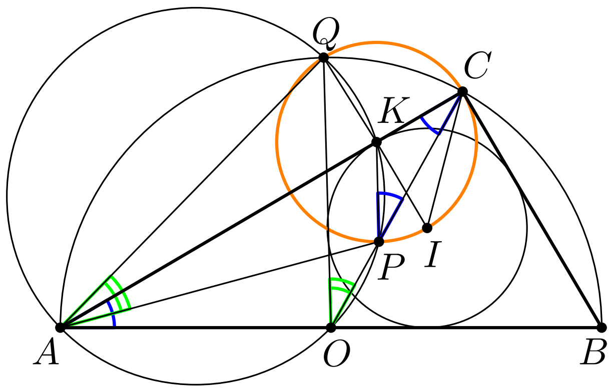

Fig. 1.

2. Let O and H be the circumcenter and orthocenter of the acute triangle ABC. On sides AC and AB, points D and E are chosen respectively such that segment DE passes through point O and $DE \parallel BC$. On side BC, points X and Y are chosen such that $BX = OD$ and $CY = OE$. Prove that $\angle XHY + 2\angle BAC = 180°$.

(*Matthew Kurskyi*)

Solution 1. We will show that $HY = CY$. To do so, construct the perpendiculars $OM \perp AB$ and $YN \perp CH$ (Fig. 2). Since O is the orthocenter of the triangle formed by the midlines of triangle ABC, we have $CH \parallel OM$ and $CH = 2OM$. The right triangles OME and CNY are congruent by one leg and an acute angle ($OE = CY$ by condition, and $\angle MOE = \angle NCY$ since their corresponding sides are parallel). Therefore, $CN = OM = \frac{1}{2}CH$. Hence, in triangle CYH, the height YN is also a median, so $HY = CY$. Therefore,

$$\angle HYX = 2\angle HCB = 2(90^\circ - \angle ABC) = 180^\circ - 2\angle ABC.$$

Similarly, $HX = XB$, so $\angle HXY = 180^\circ - 2\angle ACB$. Thus,

$$\angle XHY = 180^\circ - \angle HYX - \angle HXY = 2\angle ABC + 2\angle ACB - 180^\circ = 180^\circ - 2\angle BAC.$$

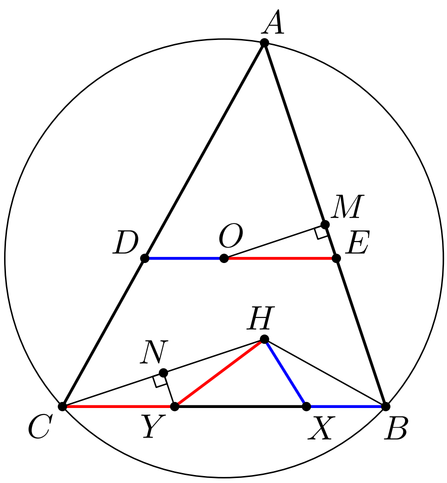

Fig. 2.

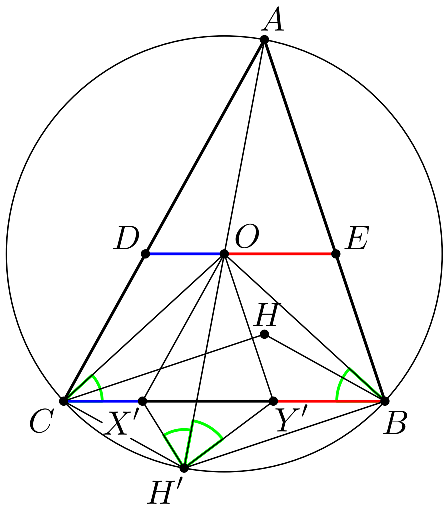

Fig. 3.

Solution 2. Let H', X', and Y' be the reflections of H, X, and Y with respect to the midpoint of side BC (Fig. 3). Since $CH \perp AB$, $BH \perp AC$, and $BHCH'$ forms a parallelogram, it follows that $\angle ABH' = \angle ACH' = 90^\circ$, so AH' is a diameter of the circumcircle of triangle ABC. Since $CX' = BX = OD$ and $CX' \parallel OD$, quadrilateral $ODCX'$ is a parallelogram. Therefore, $OX' \parallel AC$, which implies $OX' \perp CH'$. Thus, point X' lies on the altitude of the isosceles triangle OCH' ($OC = OH'$ as radii), so

$$\angle OH'X' = \angle OCB = \frac{1}{2}(180^\circ - \angle BOC) = 90^\circ - \angle BAC.$$

Similarly, $\angle OH'Y' = 90^\circ - \angle BAC$, so

$$\angle XHY = \angle X'H'Y' = \angle OH'X' + \angle OH'Y' = 180^\circ - 2\angle BAC.$$

3. Inside triangle ABC, points D and E are chosen such that $\angle ABD = \angle CBE$ and $\angle ACD = \angle BCE$. Point F on side AB is such that $DF \parallel AC$, and point G on side AC is such that $EG \parallel AB$. Prove that $\angle BFG = \angle BDC$.

(*Anton Trygub*)

Solution. Let the rays CD and BE intersect the circumcircle of triangle ABC at points P and Q, respectively, and let segment PQ intersect sides AB and BC at points F' and G', respectively (Fig. 4). We will prove that $F' = F$ and $G' = G$.

Since $\angle DPF' = \angle CPQ = \angle CBQ = \angle DBF'$, the points B, P, F', D lie on a circle. It follows that

$$\angle BF'D = \angle BPC = \angle BAC.$$

Thus, $DF' \parallel AC$, which implies $F' = F$. Similarly, we have $G' = G$.

From this, it follows that triangles BFQ and BDC are similar because $\angle FBQ = \angle DBC$ and $\angle FQB = \angle DCB$. Therefore,

$$\angle BFQ = \angle BDC.$$

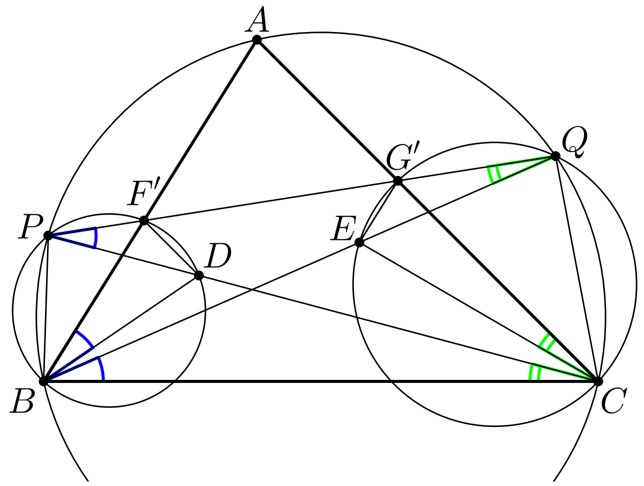

Fig. 4.

4. Let I and M be the incenter and the centroid of a scalene triangle ABC, respectively. A line passing through point I parallel to BC intersects AC and AB at points E and F, respectively. Reconstruct triangle ABC given only the marked points E, F, I, and M.

(*Hryhorii Filippovskyi*)

Solution. Let T be the midpoint of EF. Consider two cases.

Case 1. $T \neq M$. Let Q be the intersection of the external bisector of $\angle BAC$ with line EF (Fig. 5). By the angle bisector theorem and the exterior angle bisector theorem, we have

$$EQ : FQ = AE : AF = EI : FI,$$

so point Q can be constructed. Since line TM contains the median AD and $\angle QAI = 90°$, point A is found as the intersection of TM and the semicircle with diameter QI.

Case 2. $T = M$. The point I divides the bisector AL in the ratio

$$AI : IL = AM : MD = 2 : 1 = (AB + AC) : BC = (AF + AE) : FE.$$

Thus, $AF = 2FI$ and $AE = 2EI$, meaning that A is the intersection of circles centered at E and F with radii $2EI$ and $2FI$, respectively.

Once point A is determined, construct point D such that $AD = \frac{3}{2}AM$, and draw a line through D parallel to EF. This line intersects rays AF and AE at points B and C, respectively.

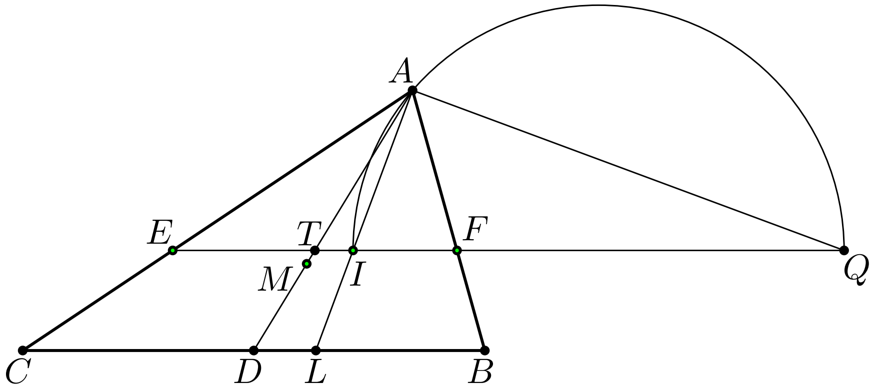

Fig. 5.

Note. The reconstruction of point A is generally not unique. The conditions may be satisfied by one or both of the constructed triangles ABC.

5. Let $ABCDEF$ be a cyclic hexagon such that $AD \parallel EF$. Points X and Y are marked on diagonals AE and DF, respectively, such that $CX = EX$ and $BY = FY$. Let O be the intersection point of AE and FD, P the intersection point of CX and BY, and Q the intersection point of BF and CE. Prove that points O, P, and Q are collinear.

(*Matthew Kurskyi*)

Solution. Let K and L be the intersection points of BF and CE with AD, and let ω_1 and ω_2 be the circumcircles of triangles AKB and DLC, respectively (Fig. 6). We will show that points O, P, Q lie on the radical axis of circles ω_1 and ω_2.

Since $AFED$ is an isosceles trapezoid, we have $\angle ABF = \angle ADF = \angle DAE = \angle DCE$. Therefore, AO and DO are tangents to circles ω_1 and ω_2, respectively, and $AO = DO$. Hence, point O lies on the radical axis of circles ω_1 and ω_2.

Since $\angle YBF = \angle BFD = \angle BAD$, it follows that BP is tangent to circle ω_1. Similarly, CP is tangent to circle ω_2. To show that $\angle PBC = \angle PCB$, we observe that

$$\angle PBC = \angle FBC - \angle BFD = \\ = \frac{1}{2}(\smile FDC - \smile BCD) = \frac{1}{2}(\smile FED - \smile BC),$$

and similarly,

$$\angle PCB = \angle ECB - \angle AEC = \frac{1}{2}(\smile AFE - \smile BC).$$

Since $\smile FED = \smile AFE$ due to $AD \parallel EF$, we conclude $PB = PC$, so point P lies on the radical axis of circles ω_1 and ω_2.

Finally, points K, L, C, and B are concyclic because $\angle BKL = \angle BFE = 180° - \angle BCE$. Let this circle be ω_3. The lines BF and CE are the radical axes of circles ω_1, ω_3 and ω_2, ω_3, respectively. Therefore, point Q is the radical center of circles $\omega_1, \omega_2, \omega_3$, and thus lies on the radical axis of circles ω_1 and ω_2.

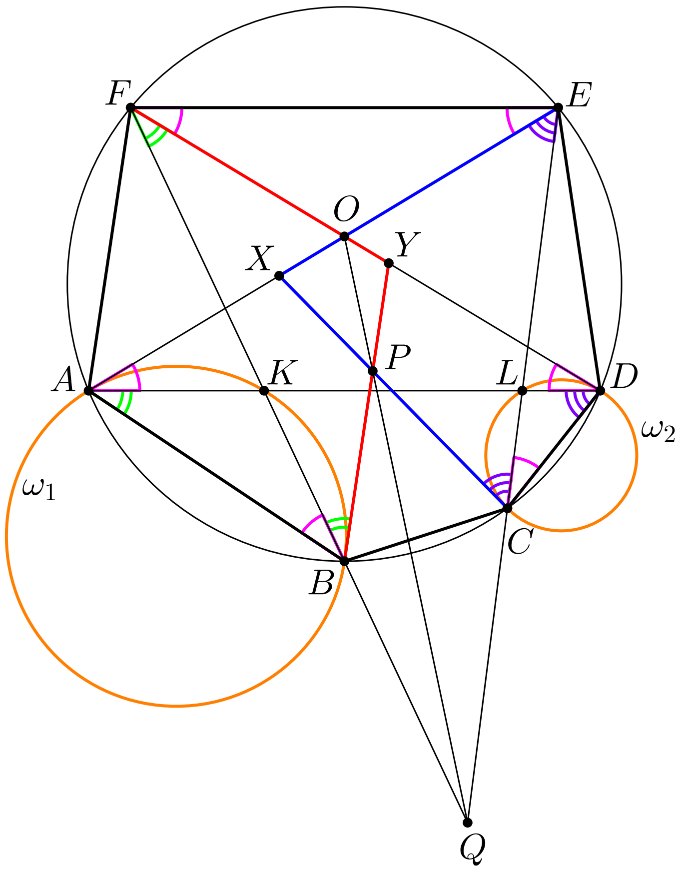

Fig. 6.