Problems and Solutions

8th Grade

1. Given a triangle ABC, a point D is chosen on the side BC, and a point E is chosen inside the triangle such that $\angle BAD = \angle ECD$ and $\angle DEC = \angle ABC$. Prove that $\angle BEC = 180° - \angle BAC$.

(*Heorhii Zhilinskyi*)

Solution 1. Let line AD meet the circumcircle of triangle ABC again at a point F (Fig. 1). Then $\angle BCF = \angle BAF = \angle ECD$ and $\angle AFC = \angle ABC = \angle DEC$. Hence triangles DCE and DCF have two pairs of equal angles, so their third angles are also equal. It follows that triangles DCE and DCF are congruent by the ASA theorem, therefore $CE = CF$. Thus triangles BCE and BCF are congruent by the SAS theorem, so $\angle BEC = \angle BFC = 180° - \angle BAC$.

Solution 2. Let line CE meet side AB at a point G (Fig. 2). Since $\angle GAD = \angle GCD$, quadrilateral $GACD$ is cyclic too. Moreover

$$\angle GED = 180° - \angle DEC = 180° - \angle GBD,$$

so quadrilateral $GBDE$ is cyclic. Therefore

$$180° - \angle BEC = \angle GEB = \angle GDB = 180° - \angle GDC = \angle BAC.$$

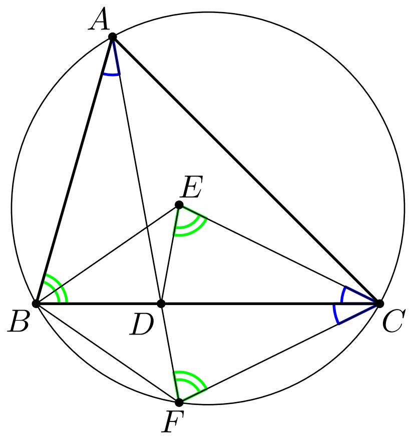

Fig. 1.

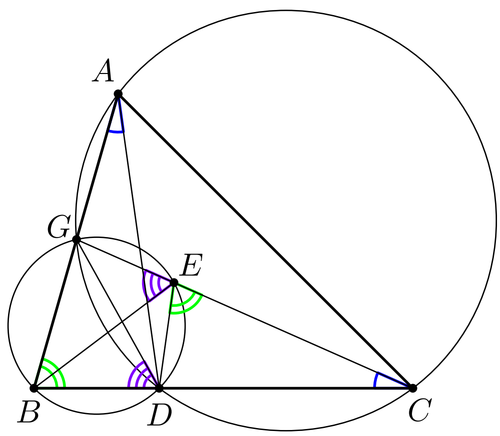

Fig. 2.

2. On the side AC of triangle ABC, a point D is chosen such that $BD = CD$, and on the segment BD a point E is chosen such that $CE = AB$. Suppose that $AB + BE = AC$. Find $\angle BAC$. (*Heorhii Zhilinskyi*)

Solution. Triangle BDC is isosceles, therefore $\angle DBC = \angle DCB$. Choose a point F on the side AC such that $CF = BE$ (Fig. 3). Triangles BCE and CBF are congruent by the SAS theorem, hence $BF = CE = AB$. Moreover

$$AF = AC - CF = AC - BE = AB.$$

Thus triangle ABF is equilateral, so $\angle BAC = 60°$.

Answer: $\angle BAC = 60°$.

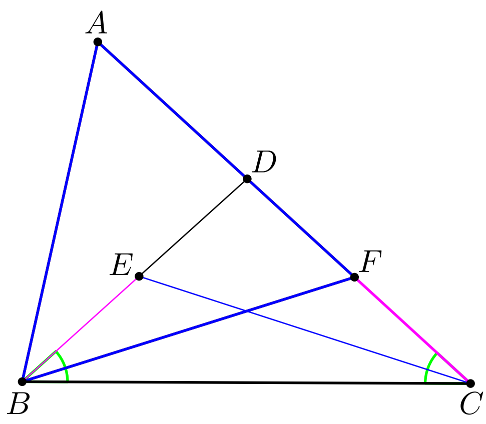

Fig. 3.

3. Let M be the midpoint of the side BC of a triangle ABC, and let P and Q be the midpoints of the altitudes BE and CF respectively. Reconstruct the triangle ABC, given only marked points M, P, and Q. (*Hryhorii Filippovskyi*)

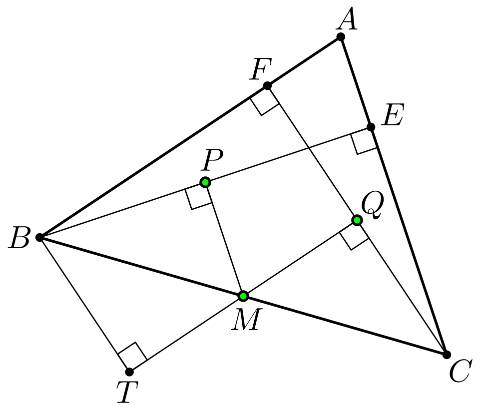

Fig. 4.

Solution. Since MP and MQ are the midlines of triangles BEC and BFC, we have $MP \parallel CE$ and $MQ \parallel BF$, so $\angle BPM = \angle CQM = 90°$. Consider a point T such that M is the midpoint of segment TQ (Fig. 4). Triangles BMT and CMQ are congruent by the SAS theorem, hence $\angle BTM = \angle CQM = 90°$. Thus the triangle can be reconstructed as follows:

1) Extend segment QM beyond M and mark a point T such that $MT = QM$.

2) Erect a perpendicular at the point P to MP and another at the point T to MT. These lines intersect at point B.

3) Extend line BM beyond M and mark point C such that $MC = BM$.

4) Drop perpendiculars from B to CQ and from C to BP, they intersect at point A.

4. Let O be the circumcenter of an acute triangle ABC. Points D and E are chosen on the sides AB and AC respectively so that segment DE passes through point O. Let K and L be the orthocenters of triangles BOD and COE respectively, and let T be the intersection point of lines KD and LE. Prove that the points A, K, T, and L are concyclic. (*Matthew Kurskyi*)

Solution. Triangle AOB is isosceles, so $\angle OAD = \angle OBD$. Due to $OK \perp BD$ and $KD \perp BO$, we have $\angle OKD = \angle OBD$. Thus $\angle OAD = \angle OKD$, therefore points O, A, K, and D are concyclic (Fig. 5). Similarly, points O, A, L, and E are concyclic. Therefore we have[1]

$$\angle KAL = \angle KAO + \angle LAO = \angle TDO + \angle TEO = 180° - \angle DTE = 180° - \angle KTL,$$

whence the quadrilateral $AKTL$ is cyclic.

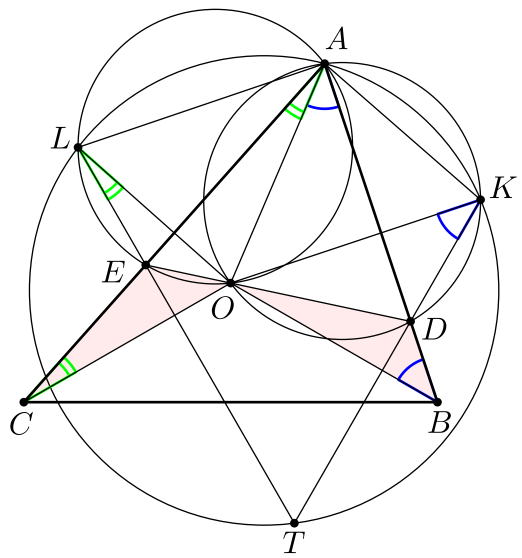

Fig. 5.

[1]for the configuration shown in Fig. 5; in other cases reasonings are quite similar.

5. Let ABC be an acute triangle with orthocenter H and circumcenter O. Suppose that there exists a point P on the side BC such that $OP = OH$ and $HP = AH$. Prove that the point P lies on line AO or on line AH. (*Mykhailo Sydorenko*)

Solution. On the extension of the altitude AD beyond point D, choose a point N such that $DN = HD$. The line BC is the perpendicular bisector of segment HN, hence $NP = HP = AH$ and $CH = CN$. Since $\angle DHC = \angle ABC$ (the angle between the altitudes of the triangle), we have $\angle ANC = \angle DHC = \angle ABC$, so the point N lies on the circumcircle of the triangle ABC. Therefore $ON = OA$ as radii, also we have $OP = OH$ and $NP = AH$. Thus triangles ONP and OAH are congruent by the SSS theorem. Denote $\angle OAH = \angle ONA = \alpha$. Then $\angle ONP = \angle OAH = \alpha$.

If the points P and A lie on the same side of line ON, it follows that P lies on the ray NA (Fig. 6a).

Now assume that the points P and A lie on the opposite sides of the line ON (Fig. 6b). In this case we have $\angle PHN = \angle PNH = 2\alpha$. Hence the base angle of isosceles triangle AHP equals $\angle PAH = \alpha$, therefore P lies on the ray AO.

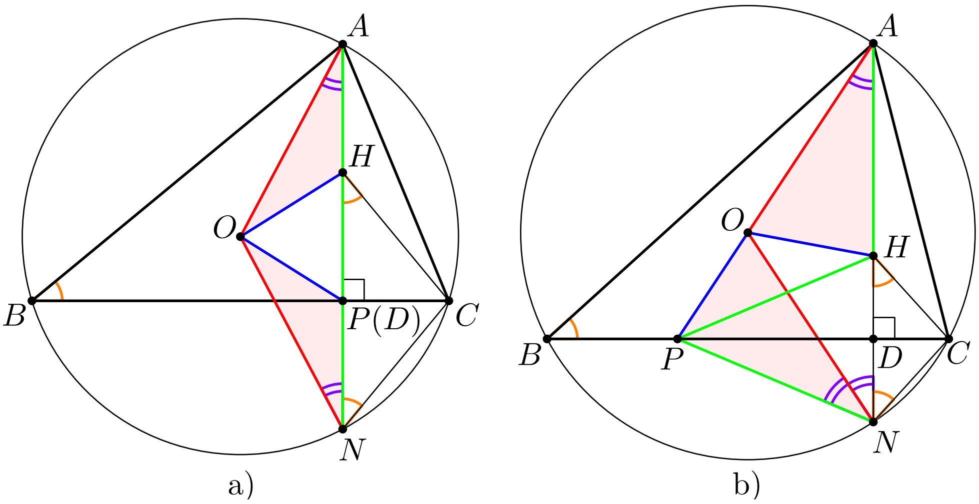

Fig. 6.

9th Grade

1. Let $ABCD$ be a cyclic quadrilateral. On the side AD, there exist points K and L such that $AK = BK$ and $CL = DL$, moreover points A, K, L, D lie on line AD in this order. Let M be a point such that $KM \parallel AB$ and $LM \parallel CD$. Prove that $BM = CM$. (*Matthew Kurskyi*)

Solution. Let O be the circumcenter of the quadrilateral $ABCD$, and let OE and OG be the perpendicular bisectors of the sides AB and CD respectively. Denote F the intersection point of line OM with BC (Fig. 1). Since the points K and L lie on lines OE and OG, and $KM \parallel AB$ and $LM \parallel CD$, we have $\angle OKM = \angle OLM = 90°$. Hence quadrilateral $OKML$ is cyclic.

Denote $\angle BAD = \alpha$. Then $\angle MKL = \alpha$ because $KM \parallel AB$, and $\angle MOL = \angle MKL = \alpha$ since quadrilateral $OKML$ is cyclic. Now in quadrilateral $FOGC$ we have $\angle FOG = \alpha$, $\angle FCG = \angle BCD = 180° - \alpha$, and $\angle OGC = 90°$. Hence $\angle OFC = 90°$. Thus line OF is the perpendicular bisector of BC, and since point M lies on OF, it follows that $BM = CM$.

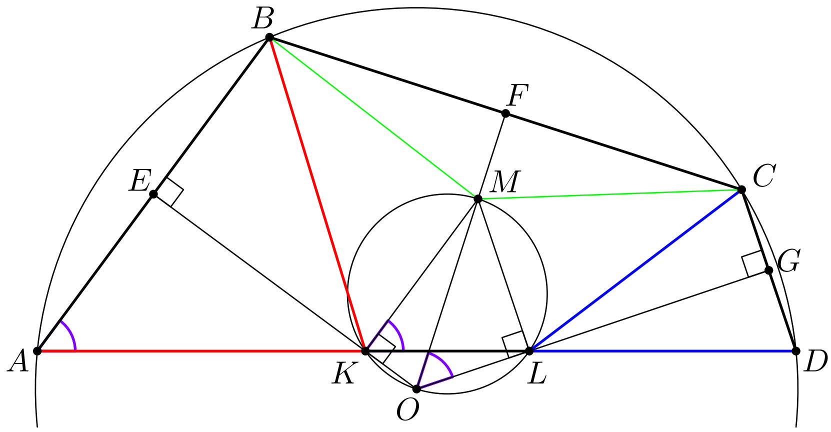

Fig. 1.

2. In a triangle ABC, points P and Q are chosen on rays AC and BC respectively, so that the circumcircles of triangles ACQ and BCP are tangent to the line AB. Let O be the circumcenter of triangle PCQ. Prove that $AO = BO$.

(*Volodymyr Pryhunov*)

Solution. Denote R the circumradius of the triangle PCQ. By the tangent–secant theorem, we have (Fig. 2)

$$(AO - R)(AO + R) = AC \cdot AP = AB^2,$$
$$(BO - R)(BO + R) = BC \cdot BQ = AB^2.$$

Hence $AO^2 - R^2 = BO^2 - R^2$, therefore $AO = BO$.

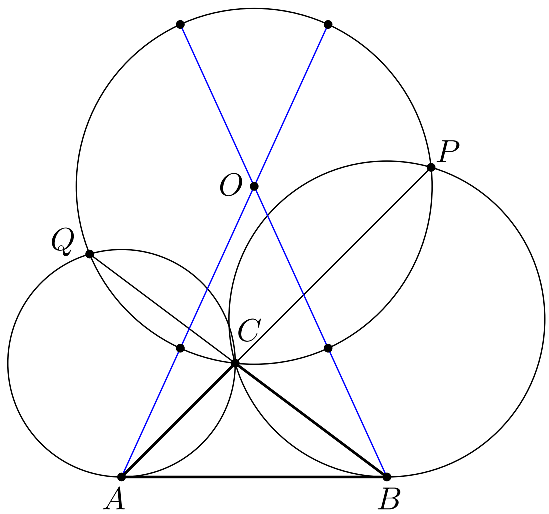

Fig. 2.

3. Let BE and CF be the angle bisectors of triangle ABC. On the extension of line EF beyond F, a point P is chosen such that $AB = BP$, and on the extension of line FE beyond E, a point Q is chosen such that $AC = CQ$. Prove that $\angle BPQ = \angle CQP$. (*Heorhii Zhilinskyi*)

Solution. Let K and L be points on line EF such that $AK \parallel BP$ and $AL \parallel CQ$. (Fig. 3). Denote $BC = a$, $AC = b$, and $AB = c$. By the angle bisector theorem, $AF/FB = b/a$. Since triangles PBF and KAF are similar, $AK/BP = AF/BF = b/a$, hence $AK = BP \cdot \frac{b}{a} = \frac{bc}{a}$.

Similarly, $AL = \frac{bc}{a}$. Therefore triangle AKL is isosceles, thus

$$\angle BPQ = \angle AKL = \angle ALK = \angle CQP.$$

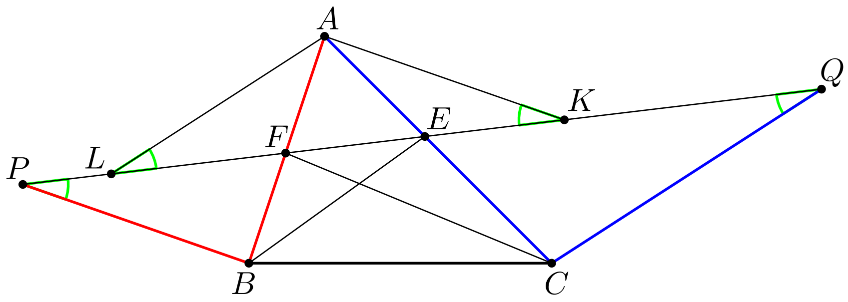

Fig. 3.

4. Let $ABCD$ be a cyclic quadrilateral with $AD \nparallel BC$. Points X and Y are chosen on the sides AB and CD respectively such that $AX/XB = CY/YD$. Points P and Q are the reflections of the point X with respect to the lines AD and BC respectively. Prove that $PY = QY$.

(*Heorhii Zhilinskyi*)

Solution. Let the extensions of the lines AD and BC meet at a point O. Since AD and BC are the perpendicular bisectors of segments XP and XQ, the point O is the circumcenter of triangle PXQ (Fig. 4). Denote Y' the intersection point of the perpendicular bisector of PQ with the line CD. Let us prove that $Y' = Y$. Then it will follow that $PY = QY$.

Since OA, OB, and OY' are the angle bisectors of angles POX, XOQ, and POQ respectively, the ray OY' lies between OA and OB, hence the point Y' lies on the segment CD. Triangles OAB and OCD are similar. Let us show that $\angle AOX = \angle COY'$. Indeed, $\angle AOX = \frac{1}{2}\angle POX = \angle PQX$, and since $PQ \perp OY'$ and $QX \perp OC$, it follows that $\angle COY' = \angle PQX = \angle AOX$. Therefore X and Y' are corresponding points in the similar triangles OAB and OCD, so $CY'/Y'D = AX/XB = CY/YD$. Hence $Y' = Y$.

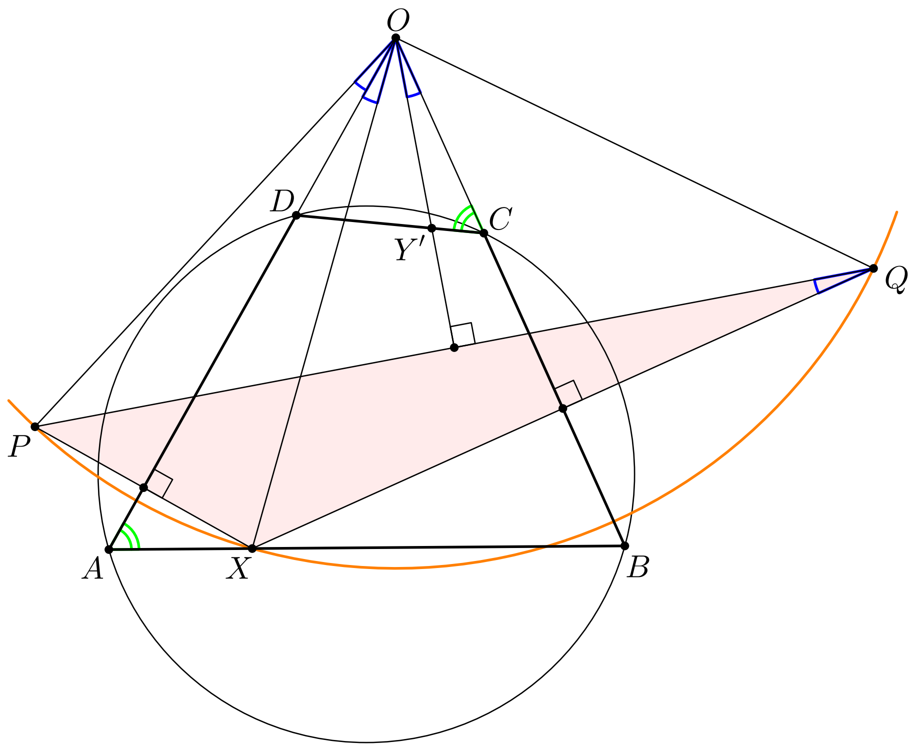

Fig. 4.

5. Let O be the circumcenter of an isosceles triangle ABC $(AB = AC)$, K be the midpoint of the arc AB of the circumcircle which does not contain the point C, T be the point on line BO such that $\angle KAT = 90^\circ$, and E be the midpoint of AC. Prove that $\angle KET = 90^\circ$.

(*Pavlo Protsenko and Anhelina Shkurynska*)

Solution. Let D be the point on the line BO such that $KD \perp BO$ (Fig. 5). Then the points K, A, T, and D lie on a circle with diameter KT. We will show that the point E also lies on this circle, whence $\angle KET = 90^\circ$.

Since OK is the angle bisector of $\angle AOB$, we have $\angle KOB = \frac{1}{2}\angle AOB = \angle ACB < 90^\circ$. Thus the point D lies on the ray OB and $\angle KOD = \angle KOB = \frac{1}{2}\angle AOB$. Similarly, OE is the angle bisector of $\angle AOC$, therefore $\angle AOE = \frac{1}{2}\angle AOC$. Hence $\angle KOD = \angle AOE$. Since $OK = OA$ as radii, the right triangles KOD and AOE are congruent by the hypotenuse and an acute angle. Therefore $KD = AE$ and $\angle DKO = \angle EAO$. Triangle KOA is isosceles, so $\angle OKA = \angle OAK$. It follows that $\angle DKA = \angle DKO + \angle OKA = \angle EAO + \angle OAK = \angle EAK$. Thus quadrilateral $DKAE$ is an isosceles trapezoid. Therefore point E lies on the circumcircle of triangle KAD, which completes the proof.

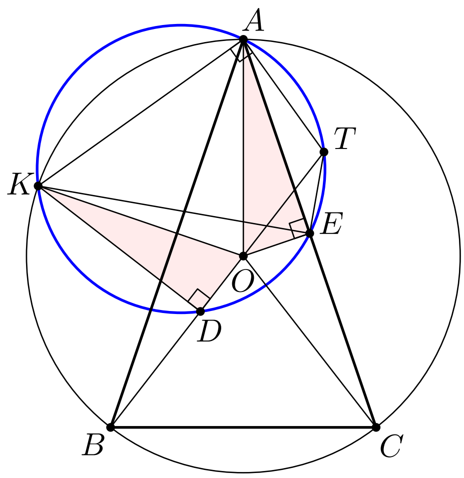

Fig. 5.

10 – 11th Grade

1. Let ABC be a triangle with $AB = AC = 2BC$. Denote ω its incircle and I its incenter. The circle ω is tangent to the side AC at a point K, and F is the second point of intersection of line BK with ω. Prove that the points A, I, F, and B are concyclic.

(*Matthew Kurskyi*)

Solution. Since $KC = \frac{1}{2}(AC + BC - AB) = \frac{1}{2}BC$, we have $KC/BC = BC/AC$, therefore triangles ABC and BKC are similar (Fig. 1). Let $\angle BAC = 2\alpha$. Then

$$\angle CKB = \angle ABC = 90° - \alpha.$$

Since $\angle IKC = 90°$, we have

$$\angle IKF = \angle IKC - \angle CKB = 90° - (90° - \alpha) = \alpha.$$

Triangle IKF is isosceles, hence $\angle IFK = \angle IKF = \alpha$. Therefore $\angle IAB + \angle IFB = \alpha + (180° - \alpha) = 180°$, so quadrilateral $AIFB$ is cyclic.

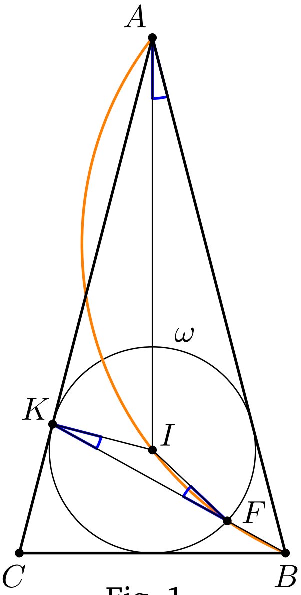

Fig. 1.

2. Let O be the circumcenter of an acute triangle ABC. On the sides AB and AC, points K and L are chosen respectively, such that $OK = BK$ and $OL = CL$. The circumcircles of triangles ABC and AKL meet again at a point T. Prove that $AT \parallel BC$. (*Matthew Kurskyi*)

Solution. Denote R the circumradius of the triangle ABC. Since triangles AOB and BOK are isosceles and have a common angle at vertex B, they are similar. Hence $AB/BO = BO/BK$, i. e., $AB \cdot BK = BO^2 = R^2$. Similarly, $AC \cdot CL = R^2$.

Choose a point D such that $ABCD$ is an isosceles trapezoid ($AD \parallel BC$), and a point E on the side CD such that $CE = BK$ (Fig. 2). Since $CD \cdot CE = AB \cdot BK = AC \cdot CL$, the points D, E, A, and L lie on the same circle. Since $ADEK$ is an isosceles trapezoid, this circle also passes through point K, so it coincides with the circumcircle of the triangle AKL. Therefore $T = D$, thus $AT \parallel BC$.

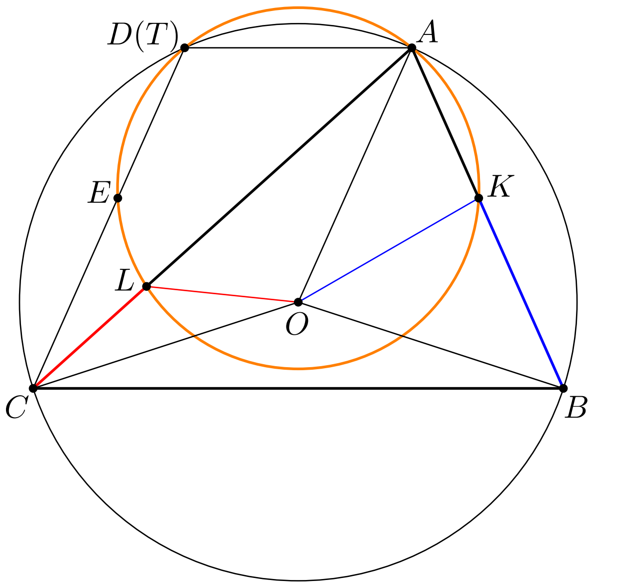

Fig. 2.

3. Let $ABCD$ be a trapezoid ($AD \parallel BC$). A point K is chosen on the side CD. Circles with centers I and J are inscribed in triangles BCK and ADK respectively. Find all trapezoids $ABCD$ for which it might happen that both polygons $ABIKJ$ and $DCIJ$ are cyclic.

(*Volodymyr Brayman and Alexandr Tolesnikov*)

Solution. Denote the angles of the trapezoid $\angle BCD = \gamma$ and $\angle CDA = \delta$, $\gamma + \delta = 180^\circ$. Then $\angle BIK = 90^\circ + \frac{\gamma}{2}$ and $\angle AJK = 90^\circ + \frac{\delta}{2}$. Since the pentagon $ABIKJ$ is cyclic, we have

$$\angle BAK = 180^\circ - \angle BIK = 90^\circ - \tfrac{\gamma}{2} \quad \text{and} \quad \angle ABK = 180^\circ - \angle AJK = 90^\circ - \tfrac{\delta}{2}.$$

Hence $\angle BAK + \angle ABK = 180^\circ - \frac{\gamma+\delta}{2} = 90^\circ$, so ABK is a right triangle.

Since $\angle KIC = 90^\circ + \frac{1}{2}\angle CBK$ and $\angle JIK = \angle JAK = \frac{1}{2}\angle DAK$, we have

$$\angle JIC = \angle KIC + \angle JIK = 90^\circ + \tfrac{1}{2}(\angle CBK + \angle DAK).$$

Note that $\angle CBK + \angle DAK = 180^\circ - (\angle ABK + \angle BAK) = 90^\circ$. So $\angle JIC = 135^\circ$. Since quadrilateral $DCIJ$ is cyclic, we get $\angle CDJ = \frac{\delta}{2} = 45^\circ$. Hence $\gamma = \delta = 90^\circ$. Thus $ABCD$ is a right trapezoid.

Since $\angle JIC = 135^\circ$ and $\angle ICD = \frac{\gamma}{2} = 45^\circ$, the cyclic quadrilateral $DCIJ$ is an isosceles trapezoid, hence $CI = DJ$. The right triangles BCK and KDA are similar, because $\angle KBC = \angle AKD$ as angles with mutually perpendicular sides. Since CI and DJ are corresponding segments in these triangles, the triangles are in fact congruent. Therefore

$$CD = CK + KD = AD + BC.$$

Now let us show that any trapezoid $ABCD$ with $\angle C = \angle D = 90^\circ$ and $CD = AD + BC$ is valid. Choose a point K on the side CD such that $CK = AD$ and $KD = BC$ (Fig. 3). The right triangles BCK and KDA are congruent by two legs, therefore $CI = DJ$ and triangle ABK is a right isosceles triangle. Since $\angle ICD = \angle CDJ = 45^\circ$, quadrilateral $DCIJ$ is an isosceles trapezoid, so it is cyclic. Moreover $\angle BAK = \angle ABK = 45^\circ$ and $\angle BIK = \angle AJK = 135^\circ$, so quadrilaterals $ABIK$ and $ABKJ$ are cyclic, which implies that the pentagon $ABIKJ$ is cyclic.

Answer: All trapezoids $ABCD$ with $\angle C = \angle D = 90^\circ$ and $CD = AD + BC$.

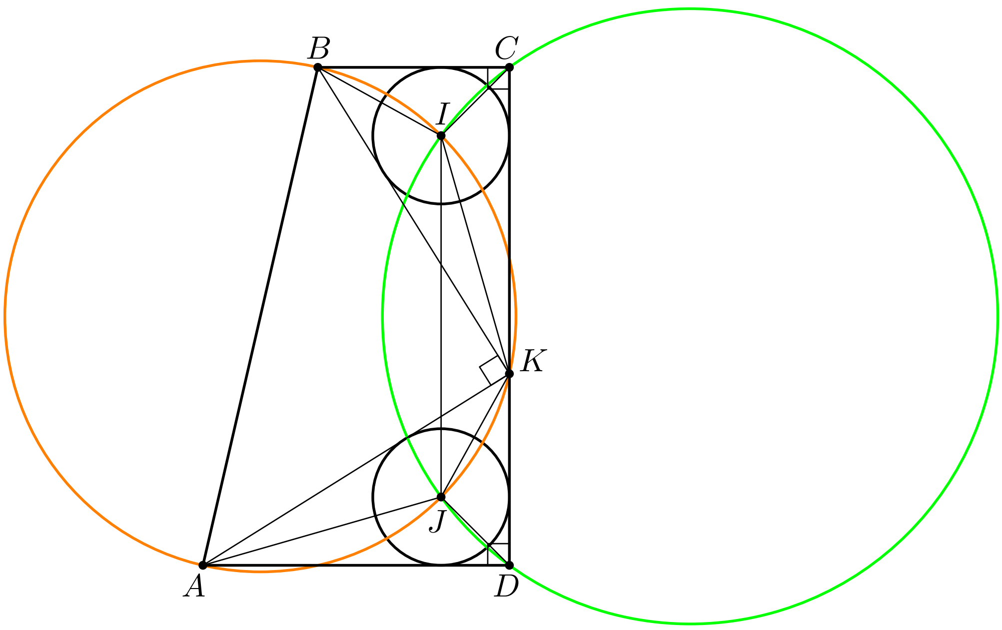

Fig. 3.

4. Let ABC be a triangle with $AB \neq AC$. Points D, E, and F are chosen on the sides BC, AC, and AB respectively, such that quadrilateral $BFEC$ is cyclic, and the circumcircle of triangle DEF is tangent to BC at the point D. On line AD, there exists a point Q such that $BQ = CQ$, moreover the points A and Q are on different sides of the line BC. Prove that

$$\angle BAC + \angle EDF + \angle BQC = 180^\circ.$$

(*Anton Trygub*)

Solution. Since $\angle BFE = 180^\circ - \angle ECB$ and $\angle EFD = \angle EDC$, we have

$$\angle BFD = \angle BFE - \angle EFD = 180^\circ - \angle ECD - \angle EDC = \angle DEC.$$

Denote $\angle BFD = \angle DEC = \alpha$. Then

$$\angle BFD = \angle BAD + \angle ADF = \alpha \text{ and } \angle DEC = \angle DAC + \angle ADE = \alpha,$$

hence $\angle BAC + \angle EDF = 2\alpha$. It remains to show that $\angle BQC = 180^\circ - 2\alpha$. Let the circumcircle of triangle BFD meet the line AD again at a point P. Then

$$AE \cdot AC = AF \cdot AB = AP \cdot AD,$$

so the circumcircle of triangle DEC also passes through the point P.

If A and P lie on the same side of BC, then $\angle BPD = \angle BFD = \alpha$ and $\angle DPC = \angle DEC = \alpha$ (Fig. 4). Thus the line AD contains the angle bisector of $\angle BPC$. Let this line meet the circumcircle of triangle BPC again at a point W. Then W is the midpoint of arc $\smile BWC$, so $BW = CW$, and $\angle BWC = 180^\circ - \angle BPC = 180^\circ - 2\alpha$. It remains to note that $Q = W$, because otherwise the line AD would be the perpendicular bisector of BC, which contradicts $AB \neq AC$.

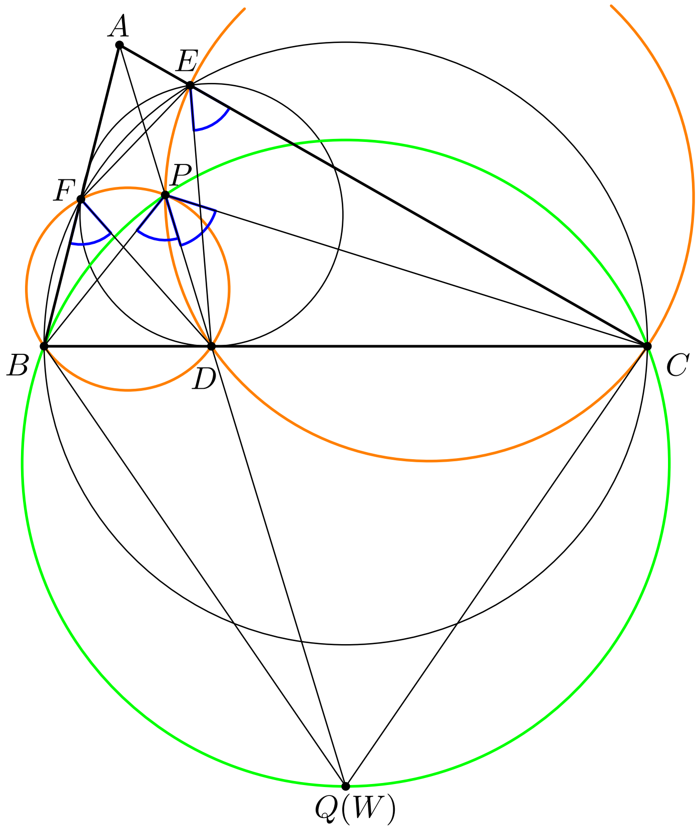

Fig. 4.

If A and P lie on the opposite sides of BC, then we obtain similarly that Q is the second intersection point of the line AD with the circumcircle of the triangle BPC, but in this case the points A and Q lie on the same side of BC, we got a contradiction.

Finally, if $P = D$, then the line AD is a common tangent to the circumcircles of the triangles BFD and DEC. Hence $\angle BDQ = \angle BFD = \alpha$, $\angle CDQ = \angle DEC = \alpha$. Thus $\alpha = 90^\circ$ and $DQ \perp BC$. Since $BW = CW$, the line $A - D - W$ is the perpendicular bisector of BC, which contradicts $AB \neq AC$.

5. Inside an acute scalene triangle ABC, a point D is chosen such that $\angle ABD = \angle ACD$. Circle with diameter AD meets the circumcircle of the triangle ABC again at a point K and the altitude AH at a point E. Prove that line KE passes through the midpoint of the side BC.

(*Mykhailo Barkulov*)

Solution. For definiteness assume that the configuration corresponds to Fig. 5. Let M be the midpoint of BC, and let the circle with diameter AD intersect the sides AB and AC at points P and Q respectively. Then BPD and CQD are right triangles. Since $\angle PBD = \angle QCD$, these triangles are similar, so $PD/QD = BP/CQ$.

Triangles KPB and KQC are also similar because $\angle KBP = \angle KBA = \angle KCA = \angle KCQ$ and $\angle KPB = 180^\circ - \angle KPA = 180^\circ - \angle KQA = \angle KQC$. It follows that $KP/KQ = BP/CQ = KB/KC$ and $\angle PKQ = \angle BKC$, so triangles PKQ and BKC are similar as well.

Draw in the circle with diameter AD a chord $DL \parallel PQ$, and let G be the intersection point of the diagonals of quadrangle $KQLP$. Let us show that G is the midpoint of PQ. Indeed, $PD = QL$ and $QD = PL$, therefore $QL/PL = PD/QD = BP/CQ = KP/KQ$, i. e., $KP \cdot PL = KQ \cdot QL$. Note that $\angle KQL = 180^\circ - \angle KPL$, so the areas of triangles KPL and KQL are equal:

$$[KPL] = \tfrac{1}{2} KP \cdot PL \sin \angle KPL = \tfrac{1}{2} KQ \cdot QL \sin \angle KQL = [KQL].$$

The triangles KPL and KQL share the side KL and have equal areas, so the points P and Q are equidistant from the line KL and lie on the opposite sides of it, implying that the midpoint of PQ lies on KL. Thus G is the midpoint of PQ.

Since the triangles PKQ and BKC are similar, and KG and KM are their medians, we have $\angle KGP = \angle KMB$. Also $\angle KGP = \frac{1}{2}(\smile KP + \smile QL) = \frac{1}{2}(\smile KP + \smile PD) = \angle KED$. Therefore $\angle KMB = \angle KED$. Since $DE \perp AH$ and $BC \perp AH$, we have $DE \parallel BC$. If line KE intersects BC at point M', then $\angle KM'B = \angle KED = \angle KMB$. Hence $M' = M$, which completes the proof.

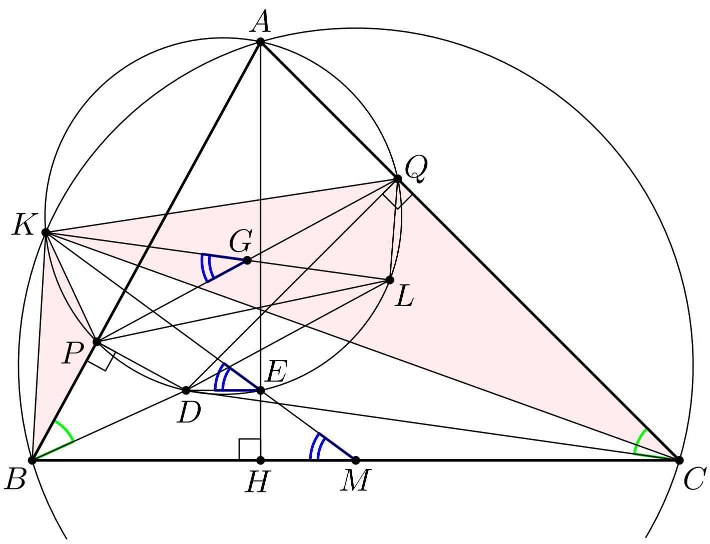

Fig. 5.